

Chapter 1

Logic, Induction And Reasoning

Logic

- Tool for mathematical inductive reasoning.

Two types:

- (i) Propositional logic (either true or false but not both)
- (ii) Predicate logic

Compound proposition: formed by combination of multiple proposition combined by connectives (logical operators)

Connectives (logical operators)

- (i) Negation ($\neg$)
- (ii) Conjunction ($\wedge$)
- (iii) Disjunction ($\vee$)
- (iv) Implication ($\rightarrow$)
- (v) Bi-implication (Bidirectional) ($\leftrightarrow$)

- (i) If p is a proposition then negation of p is denoted by $\neg p \rightarrow$ It is not the case that p .

Eg: ~~It is~~ Today is Friday

$\neg p \rightarrow$ It is not the case that today is Friday.

Truth table showing negation

p	$\neg p$
T	F
F	T

2) Conjunction $p \wedge q$ (And)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3) Disjunction $p \vee q$ (OR)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

4) Exclusive OR $p \oplus q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

5) Implication Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional proposition (Implication)

The proposition of the form "if p then q " or " p implies q " denoted by $p \rightarrow q$ is said to be conditional proposition. It is false when p is true and q is false and true otherwise.

Example: If it is sunny, then we will go for swimming.

p	q
↓	↓
(premise or antecedent or hypothesis)	(conclusion, consequence)
(sufficient condition)	(necessary)

Different ways to express conditional statements:

- If p then q
- p implies q .
- q whenever p
- q when p .
- If p , q
- necessary condition for p is q
- sufficient condition for q is p .
- p only if q .
- q if p
- q follows from p .

Converse, contrapositive and inverse of the conditional statement:

$$p \rightarrow q$$

$\neg$ = negation

$p \rightarrow q$	Contrapositive	Inverse
Converse: $q \rightarrow p$	$\neg q \rightarrow \neg p$	$\neg p \rightarrow \neg q$

Find the converse, inverse and contrapositive of the following conditional statement.

- a) If it is sunny, then we will go for swimming.

let p denotes the proposition - It is sunny.
let q denotes the proposition - we will go for swimming.

Converse

Inverse is :

$$q \rightarrow p$$

If we go for swimming then it is sunny

Contrapositive

$$\neg q \rightarrow \neg p$$

If we do not go for swimming then it is not sunny.

Inverse: $\neg p \rightarrow \neg q$

If it is not sunny, we will not go for swimming.

- b) The home team wins whenever it is raining.
 q p

Original form: $p \rightarrow q$

If it is raining then home team wins.

Intw Converse:

If the home team wins then it is raining.

Contrapositive:

If the home team does not win then it is not raining.

Inverse: $\neg p \rightarrow \neg q$

If it is ~~not~~ is not raining then home team does not win.

Show that implication and its contrapositive are equivalent.

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Q. Construct the truth table for the following compound propositions.

$$p \vee \neg q \rightarrow p \wedge q$$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$p \vee \neg q \rightarrow p \wedge q$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Find the bitwise OR, bitwise AND and bitwise XOR of the bit string.

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0110110110
1100011101
1110111111 → Bitwise OR

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0110110110
1100011101
0100010100 → Bitwise AND

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0110110110
1100011101
1010101011 → Bitwise XOR

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Bi-Conditional proposition (Bi-implication)

→ The proposition of the form "p if and only if q" denoted by $p \leftrightarrow q$ is said to be bi-conditional proposition. It is true whenever both p and q have same truth values and false otherwise.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$$

Example:

p only if q

You can take the flight if and only if you buy a ticket.

Types of compound proposition:

- 1) Tautology
- 2) Contradiction
- 3) Contingency

Tautology : A compound proposition which is always true no matter what the truth values of the proposition that occur in it is called a tautology. ($p \vee \neg p$)

p	q	$\neg p$	$p \vee \neg p$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	T

Contradiction

A compound proposition that is always false no matter what the truth values of the proposition that occur in it is called a contradiction.

Example $p \wedge \neg p$.

p	q	$\neg p$	$p \wedge \neg p$
T	T	F	F
T	F	F	F
F	T	T	F
F	F	T	F

Contingency

A compound proposition that is neither tautology nor contradiction.

eg: $(p \wedge \neg q) \rightarrow p \wedge q$

p	q	$\neg q$	$p \wedge \neg q$	$p \wedge q$	$p \wedge \neg q \rightarrow p \wedge q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	F	F	T

Logical Equivalence:

→ Compound propositions are logically equivalent if they have the same truth values in all possible cases.

Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology. Notation $p \equiv q$ denotes p & q are logically equivalent.

Sometimes, symbol $\Leftrightarrow$ is used to indicate the p & q are logically equivalent.

Q) Show that $p \rightarrow q$ & $\neg p \vee q$ are logically equivalent

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$$\therefore p \rightarrow q \equiv \neg p \vee q$$

Q) Show that $p \wedge (q \vee r)$ & $(p \wedge q) \vee (p \wedge r)$ are logically equivalent.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

$$\therefore p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Equivalence

Name

(i) $p \wedge T \equiv p$
 $p \vee F \equiv p$ Identity laws

(ii) $p \vee T \equiv T$
 $p \wedge F \equiv F$ Domination laws

$$(iii) \quad p \vee p \equiv p$$

$$p \wedge p \equiv p$$

Idempotent laws

$$(iv) \quad \neg(\neg p) \equiv p$$

Double negation laws.

$$(v) \quad p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

Commutative laws

$$(vi) \quad (p \vee q) \vee r \equiv p \vee (q \vee r)$$

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

Associative laws.

$$(vii) \quad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$$

Distributive laws.

$$(viii) \quad \neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

De-Morgan's law.

Logical Equivalence

$$(ix) \quad p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

Absorption laws

$$(x) \quad p \vee \neg p \equiv T$$

$$p \wedge \neg p \equiv F$$

Negation laws

$$(xi) \quad p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg p \rightarrow \neg q$$

$$(xii) \quad p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Q) Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing the series of logical

equivalences.

$$\begin{aligned} \textcircled{1} \neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) \quad [:: \text{De-Morgan's law}] \\ &\equiv \neg p \wedge \neg(\neg p) \vee \neg q \\ &\equiv \neg p \wedge (p \vee \neg q) \quad [:: \text{Using Double Negation law}] \\ &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \quad [:: \text{Using distributive law}] \\ &\equiv F \vee (\neg p \wedge \neg q) \quad [:: \text{Negation law}] \\ &\equiv (\neg p \wedge \neg q) \vee F \quad [:: \text{Commutative law}] \\ &\equiv \neg p \wedge \neg q \quad [:: \text{Identity law}] \end{aligned}$$

2) Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

$$\begin{aligned} (p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) \\ &\equiv \neg p \vee \neg q \vee (p \vee q) \\ &\equiv (\neg p \vee p) \vee (\neg q \vee q) \\ &\equiv T \vee T \\ &\equiv T. \end{aligned}$$

③ show that $[(p \wedge q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology.

$$[(p \wedge q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

$$p \wedge [q \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r) \quad [:: \text{Associative law}]$$

$$\begin{aligned}
&\equiv p \wedge \neg q \wedge (\neg p \vee r) \rightarrow (p \rightarrow r) \quad [E] \\
&\equiv p \wedge [(q \wedge \neg q) \vee (q \wedge r)] \rightarrow \neg p \vee r \quad [Distributive law] \\
&\equiv p \wedge [F \vee (q \wedge r)] \rightarrow (p \rightarrow r) \quad [Negation] \\
&\equiv p \wedge (q \wedge r) \rightarrow (p \rightarrow r) \\
&\equiv p \wedge q \wedge r \rightarrow (p \rightarrow r) \\
&\equiv p \wedge q \wedge r \rightarrow (\neg p \vee r) \quad [E] \\
&\equiv \neg (p \wedge q \wedge r) \vee (\neg p \vee r) \\
&\equiv \neg p \vee \neg q \vee \neg r \vee \neg p \vee r \\
&\equiv \neg p \vee \neg p \vee \neg q \vee \neg r \vee r \\
&\equiv \neg p \vee \neg p \vee \neg q \vee T \\
&\equiv T. \quad \text{So it is a tautology.}
\end{aligned}$$

Show that $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \wedge r)] \rightarrow r$ is a tautology.

$$\begin{aligned}
&[(p \vee q) \wedge (p \rightarrow r) \wedge (q \wedge r)] \rightarrow r. \\
&\equiv \neg [(p \vee q) \wedge (p \rightarrow r) \wedge (q \wedge r)] \vee r \\
&\equiv \neg (p \vee q) \vee \neg (p \rightarrow r) \vee \neg (q \wedge r) \vee r \\
&\equiv (\neg p \wedge \neg q) \vee \neg (\neg p \vee r) \vee (\neg q \vee \neg r) \vee r. \\
&\equiv (\neg p \wedge \neg q) \vee (p \wedge \neg r) \vee (\neg q \vee \neg r \vee r) \\
&\equiv (\neg p \wedge \neg q) \vee (p \wedge \neg r) \vee (T) \\
&\equiv T.
\end{aligned}$$

Precedance of logical operators

- ① $\neg$
- ② $\wedge$
- ③ $\vee$
- ④ $\rightarrow$
- ⑤ $\leftrightarrow$

Q- If $P=F$, $Q=T$, $S=T$, $R=F$ then find the truth value of

$$\textcircled{i} (S \rightarrow (P \wedge \bar{R})) \wedge (CP \rightarrow (R \vee Q)) \wedge S$$

$$\Rightarrow (T \rightarrow (F \wedge \bar{F})) \wedge (CF \rightarrow (F \vee T)) \wedge T$$

$$\Rightarrow (T \rightarrow F) \wedge (CF \rightarrow T) \wedge T$$

$$\Rightarrow F \wedge (T \wedge T)$$

$$\Rightarrow F \wedge T$$

$$\Rightarrow F$$

$$\textcircled{ii} ((P \wedge \bar{Q}) \leftrightarrow (Q \wedge R)) \rightarrow (S \vee \bar{Q})$$

$$\Rightarrow (CF \wedge \bar{T}) \leftrightarrow (T \wedge F) \rightarrow (T \vee \bar{T})$$

$$\Rightarrow (F \leftrightarrow F) \rightarrow T$$

$$\Rightarrow T \rightarrow T$$

$$\Rightarrow T$$

Translate the given English sentences into compositional logic.

- i) You can access the internet from campus if and only if you are a Computer Science major or you are not a freshman.

In the form of p if and only if q , where
 p : You can access the internet from campus
 r : You are a freshman.
 q : You are a computer science major

$$p \leftrightarrow (q \vee \neg r)$$

- ii) The automated reply cannot be send when the file system is full. q when p .

q : The automated reply can be sent
 p : The filesystem is full.

$$p \rightarrow \neg q$$

- iii) Translate to propositional logic

p : ~~ZZ~~ Grizzly bears have been seen in the area

q : Hiking is safe on the trail

r : Berries are ripe along the trail

Berries are ripe along the trail but Grizzly bears have not been seen in the area

$$\Rightarrow r \wedge \neg p$$

If berries are ripe along the trail, hiking is safe if and only if Grizzly bears have not been seen in the area.

$$r \rightarrow q \leftrightarrow \neg p$$

Predicates and quantifiers.

Predicates:

statements involving variables such as:

$$"x > 3"$$

$$"x = y + 3"$$

$$"x + y = z"$$

Predicates are the declarative sentences whose truthness or falsehood depends upon the single or more variables.

Consider the statement:

$$"x \text{ is greater than } 3"$$

Here, x is the variable (subject) and the statement " x is greater than 3" is predicate. We can denote the above statement by

$$p(x): "x \text{ is greater than } 3"$$

The statement $p(x)$ is also called the value of the propositional function p at x .

$P(x)$ becomes proposition when specific values are substituted in variable x .

Predicates are used for verification that computer programs always produce desired output (result) when given valid input.

Note: If $P(x)$ denote " x is greater than 3" then, find the truth value of $P(4)$ and $P(1)$.

For $P(4)$ the statement is true.

$P(x)$: " x is greater than 3".

$P(4)$ is obtained by substituting $x=4$:

$P(4)$: 4 is greater than 3 which is true
so the truth value is True

$P(1)$: 1 is greater than 3 which is false so the truth value is False

Proof:

Consider the simple program

temp := x

x := y

y := temp

This program interchange the values of the variable x & y .

Preconditions

→ statements that describe the valid input.

let $P(x,y)$ denotes the statement " $x=a$ and $y=b$ " where a and b are the values of the variables x & y respectively.

Post conditions:

The conditions that the output should satisfy when the program is run.

let $Q(x,y)$ denotes the statement " $x=b$ and $y=a$ ".

let $P(x,y)$ holds true i.e. $x=a$ & $y=b$ is true.

First step:

$x=a$, $temp=a$, $y=b$

2nd step:

$x=b$; $temp=a$, $y=b$.

3rd step:

$x=b$, $temp=a$, $y=a$

$\therefore Q(x,y)$ holds true when $P(x,y)$ is true.

Quantifiers.

Quantification is another way to create a propositional form of from a propositional

function.

It expresses the extend to which a predicate is true over a range of elements. The words all, some, many, ~~none~~ ^{at least} ~~none~~ & few are used in quantification.

- Type of quantifiers:
- ① Universal quantifier
 - ② Existential quantifier

under The quantifiers that tells that a predicate is true for every element is known as (on the consideration is known as universal quantifier. The symbol ' $\forall$ ' is used to represent universal quantifier. $\Rightarrow$

If $P(x)$ is a predicate.
The universal quantifier of $P(x)$ is denoted by " $\forall x P(x)$ "



"For all"

"For every"

→ $\forall x P(x)$ is true when $P(x)$ is true for every value of x .

domain → All the possible values that the variable can have.

If $P(x)$ denotes the statement:

$P(x): "x+1 > x"$. What is the truth value of the quantification?

-fication $\forall x p(x)$ where domain consists of all real numbers.

Here the domain of x is all the real number, As $p(x)$ is true for every possible values of x so, $\forall x p(x)$ is true. i.e for all x $p(x)$ is true.

Universal Quantifier

=> If the elements in the domain can be listed say $x_1, x_2, x_3, \dots, x_n$ then universal quantifier ($\forall x p(x)$) is equivalent to expression $p(x_1) \wedge p(x_2) \wedge \dots \wedge p(x_n)$.

What is the truth value for $\forall x p(x)$ where $p(x)$ is the statement " x^2 is less than 10" and the domain consists of positive number not exceeding 4.

Solⁿ:

Here, $p(x)$ is the statement " x^2 is less than 10"

$$p(x): x^2 < 10$$

The elements in the domain can be listed as $\{1, 2, 3, 4\}$

$$\text{Now, } p(1) \wedge p(2) \wedge p(3) \wedge p(4)$$

$$= T \wedge T \wedge T \wedge F$$

$$= F$$

$\therefore \forall x p(x)$ is false

Existential quantifier

→ The existential quantifier of the propositional function $p(x)$ is denoted by $\exists x.p(x)$ and read as "There exists x such that $p(x)$ ", "There is a x such that $p(x)$ ", "There is at least one x such that $p(x)$ " or "for some x $p(x)$ ". Here symbol $\exists$ is the existential quantifier. The statement $\exists x.p(x)$ is true if $p(x)$ is true for at least one value of x and false otherwise.

Let $p(x)$ be the statement " x is greater than 3". What is the truth value of quantification $\exists x.p(x)$, where the domain consists of all real numbers.

=> Here $p(x)$ denote the statement " x is greater than 3" and the domain consists ~~for~~ of all real numbers.

As the $p(x)$ is true for at least one value within the domain so the statement $\exists x.p(x)$ is true.

→ When the elements in the domain can be listed say $x_1, x_2, \dots, x_n$ then $\exists x.p(x)$ is equivalent to $p(x_1) \vee p(x_2) \vee \dots \vee p(x_n)$

What is the truth value of $\exists x.p(x)$ where $p(x)$ is the statement " x^2 is greater than 10" and the universe of discourse (domain) consists of the positive integers not exceeding 4.

SOLⁿ:

Elements in the domain = $\{1, 2, 3, 4\}$

Here $p(x)$ denote the statement " x^2 is greater than 10".

$$\begin{aligned}\text{Now, } & p(1) \vee p(2) \vee p(3) \vee p(4) \\ &= F \vee F \vee F \vee T \\ &= T\end{aligned}$$

$\therefore \exists x p(x)$ is true.

#

Binding variable:

If the quantifier is used in the variable then occurrence of this variable is said to be bound. If the variable x is not bound by the quantifier, then the occurrence of this variable is said to be free.

→ The variable in the propositional function $p(x)$ should be bound or should be set to the particular values to turn it into proposition which can be done by the combination of the universal quantifier, existential quantifier or assigning the particular value.

Eg:

bound $\exists x (x+y=1)$ Free

II Logical Equivalences involving quantifiers:

- Q. Show that $\forall x(p(x) \wedge Q(x))$ and $\forall x p(x) \wedge \forall x Q(x)$ are logically equivalent, (where same domain is used throughout)

(1)

Solⁿ:

Let $\forall x(p(x) \wedge Q(x))$ be true.

→ If a is in domain, $p(a) \wedge Q(a)$ is true.

Hence, $p(a)$ is true and $Q(a)$ is true, for every elements in the domain $\forall x p(x)$ and $\forall x Q(x)$ are both true i.e.

$\forall x p(x) \wedge \forall x Q(x)$ is true.

Let $\forall x p(x) \wedge \forall x Q(x)$ is true

→ If ' a ' is in domain $p(a)$ is true and $Q(a)$ is true.

→ This follows for every element ' a ' in the domain $(p(a) \wedge Q(a))$ is true.

→ This follows $\forall x(p(x) \wedge Q(x))$ is true.

(2)

$$\exists x(p(x) \vee Q(x)) \equiv (\exists x p(x) \vee \exists x Q(x))$$

Negating Quantified expressions:

- (1) Every student in your class has taken a course in calculus.

$\forall x p(x)$ where

$p(x) \rightarrow$ "x has taken a course in calculus".
Domain is all the students in class.

The negation of the statement can be rewritten as:

It is ~~not~~ not the case that every student in ^{this} ~~your~~ class has taken a course in calculus.

$$\neg \forall x p(x)$$

The above statement is equivalent to "There is a student in your class who has not taken a course in the calculus".
i.e.

$$\exists x \neg p(x)$$

$$\neg \forall x p(x) \equiv \exists x \neg p(x)$$

- (2) There is a student in this class who has taken a course in calculus.

$\exists x p(x)$ where

$p(x) \rightarrow$ "x has taken a course in calculus".
Domain is all students in class.

The negation of the given statement is:

It is not the case that there exist a student in this class who has taken a course in calculus.

$$\neg \exists x p(x)$$

→ The statement is equivalent to "Every student in this class has not taken a course in calculus".

$$\forall x \neg p(x)$$

$$\neg \exists x p(x) \equiv \forall x \neg p(x)$$

Q. Find the Negation of

(i) $\forall x (x^2 > 0)$ (ii) $\exists x (x^2 = 2)$

(i) $\forall x (x^2 > 0)$

Here, $p(x): x^2 > 0$.

$$\neg \forall x (x^2 > 0) \equiv \exists x (x^2 \leq 0)$$

(ii) $\exists x (x^2 = 2) \equiv \forall x \neg (x^2 = 2)$
 $\neg \exists x (x^2 = 2) \equiv \forall x (x^2 \neq 2)$

Express the given statements using predicate logic and quantifiers.

a) Every student in this class ~~are~~ has studied calculus.

where, $\forall x p(x) \Rightarrow$ "Every student"

When the domain is every student, the statement can be rewritten in notation as

where, $\forall x p(x)$
 $p(x) \rightarrow$ "Every x has studied calculus".
Domain is every student in class.

When the domain is all the people/person

For every person, if person is a student in this class then person has studied calculus.

$\rightarrow$ For every person x , if x ~~has~~ is a student in this class, then x has studied calculus.

$\forall x (p(x) \rightarrow q(x))$,
where

$p(x) \rightarrow$ " x is a student in this class".
 $q(x) \rightarrow$ " x has studied calculus".

visited

b) Some student in this class has studied Mexico
 Case I: When domain is std in class
 Case II: When domain is all the people

c) Every student in this class has visited either Canada or Mexico

Case I: Domain consists of student in class
 Case II: Domain consists of all the people

d) Consider the argument
 "All lions are fierce"
 "Some lions do not drink coffee"
 "Some fierce creatures do not drink coffee".

where Domain consists of all the creatures.

b) Some student in this class has visited Mexico
 Case I: When domain is student in class.
 Case II: When domain is all the people

Solⁿ:

Case I: When domain is student in class, the statement can be rewritten in notation as
 There exists a student x with the property that x has visited Mexico
 $\exists x p(x)$
 where $p(x) = x$ has visited Mexico.

Case II:

When domain is all the people,

→ There exist people with the property that the people is in the class and the people has visited Mexico

For some person, if person is a student in this class, then x has visited Mexico.

$$\exists p(x) \rightarrow Q(x)$$

There exist people x with the property x is in the class and x has visited Mexico.

$$\exists x (p(x) \wedge Q(x))$$

where $p(x) \rightarrow x$ is in the class

$Q(x) \rightarrow x$ has visited Mexico

⊗ c) Every student in this class has visited either Canada or Mexico.

Case I: Domain consists of student in class.

$$\forall x (C(x) \vee M(x))$$

where $C(x) \rightarrow$ "x has visited Canada"

$M(x) \rightarrow$ "x has visited Mexico."

Case II: Domain consists of all the people.

For every people x if x is in the class then x has visited either Mexico or Canada.

$$\forall x (p(x) \rightarrow C(x) \vee M(x))$$

where,

$p(x) \rightarrow$ "x is in the class"

$C(x) \rightarrow$ "x has visited Canada"

$M(x) \rightarrow$ "x has visited Mexico".

d) "All lions are fierce" } Argument
 "Some lions donot drink coffee" }
 "Some fierce creature donot drink coffee". Conclusion.

domain consists of all the people creatures.

sol:-

$P(x) \rightarrow$ "x is a lion"

$Q(x) \rightarrow$ "x is fierce"

$R(x) \rightarrow$ "x drink coffee"

$\rightarrow \forall x (P(x) \rightarrow Q(x))$

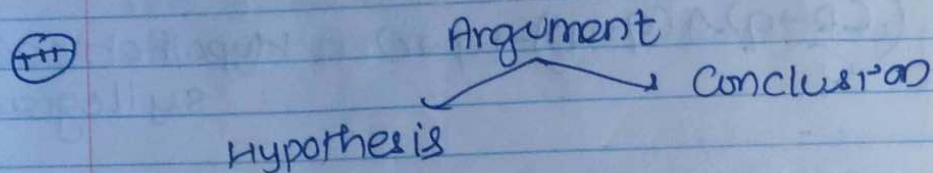
$\rightarrow$ There exists a creature such that x is a lion and x donot drink coffee.

$\exists x (P(x) \wedge \neg R(x))$

$\rightarrow \exists x (Q(x) \wedge \neg R(x))$

Note: Use $\rightarrow$ with universal quantifier
 Use $\wedge$ with existential quantifier

Rules of inference:



Rules of inference are used to deduce new statements from statements we already have. The rules of inference are the basic tools for establishing the truth of statements.

If you have a current password then you can log on to the computer network.

Soln:

$P(x)$: "You have a current password".

$Q(x)$: "You can log on to the computer network".

$$\begin{array}{lcl}
 P \rightarrow Q & \text{hypothesis} & \\
 \hline
 P & & \\
 \hline
 \therefore Q & \text{Conclusion} & [(P \rightarrow Q) \wedge P \rightarrow Q \text{ tautology}]
 \end{array}$$

Rules of inference

Tautology

Name

(1)

$$\begin{array}{lcl}
 P & & \\
 \hline
 P \rightarrow Q & & \\
 \hline
 \therefore Q & &
 \end{array}
 \quad
 \begin{array}{lcl}
 C(P) \wedge (P \rightarrow Q) & \rightarrow & Q \\
 \text{Modus ponens}
 \end{array}$$

(2)

$$\begin{array}{lcl}
 \neg Q & & \\
 \hline
 P \rightarrow Q & & \\
 \hline
 \therefore \neg P & &
 \end{array}
 \quad
 \begin{array}{lcl}
 (\neg Q \wedge (P \rightarrow Q)) & \rightarrow & \neg P \\
 \text{Modus Tollens}
 \end{array}$$

Tautology

Name _____

(iii)
$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array} \quad (Cp \rightarrow q) \wedge (Cq \rightarrow r) \rightarrow Cp \rightarrow r \quad \text{Hypothetical syllogism}$$

(iv)
$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array} \quad (Cp \vee q) \wedge (\neg p) \rightarrow q \quad \text{Disjunctive syllog}$$

(v)
$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array} \quad (p \rightarrow Cp \vee q) \quad \text{Addition.}$$

b)

(vi)
$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array} \quad (p \wedge q) \rightarrow p \quad \text{Simplification}$$

(vii)
$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array} \quad ((p \wedge q) \rightarrow (p \wedge q)) \quad \text{Conjunction}$$

(viii)
$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array} \quad ((p \vee q) \wedge (C \neg p \vee r)) \rightarrow (Cp \wedge q) \rightarrow (Cq \vee r) \quad \text{Resolution.}$$

c)

Determine the rule of inference used for the given argument.

- a) It is below freezing now. Therefore it is either below freezing or raining now.

let p denote "It is below freezing now"
let q denote "It is raining now".

$$\frac{p}{\therefore p \vee q}$$

so, the given argument follows addition rule.

b) It is below freezing and raining now. Therefore it is below freezing now

Solⁿ: let p : "It is below freezing now"
 q : "It is raining now"

$$\frac{p \wedge q}{\therefore p}$$

so, the given argument follows simplification rule.

c) If it rains today then we will not have a barbeque today. If we do not have a barbeque today then we will have a barbeque tomorrow. Therefore if it rains today, then we will have a barbeque tomorrow.

Solⁿ:

let p : "It rains today"

q : "We will have a barbeque today"

r : "We will have barbeque tomorrow"

$$\begin{array}{l} p \rightarrow \neg q \\ \neg q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

So, the given argument follows hypothetical syllogism.

show that the hypothesis
 "It is not sunny this afternoon and it is colder than yesterday". We will go for swimming only if it is sunny. If we do not go swimming then we will take a canoe trip and "If we take a canoe trip then we will be home by sunset" leads to the conclusion we will be home by sunset by using rules of inference.

Sol:-

- let p : It is sunny this afternoon.
 q : It is colder than yesterday
 r : We will go for swimming.
 s : We will take a canoe trip
 t : We will be home by sunset.

Hypothesis

$$\begin{array}{l} \neg p \wedge q \\ r \rightarrow p \\ \neg r \rightarrow s \\ s \rightarrow t \end{array}$$

Conclusion

$$t$$

Steps

Reasons

- | | |
|---|---|
| <p>① $\neg p \wedge q$
 (ii) $\neg p$</p> | <p>Hypothesis
 Simplification using ①</p> |
|---|---|

- (i) $\neg p$
- (ii) $r \rightarrow p$
- (iii) $\neg r$

using modus Tollens on
(ii) & (iii).

(iv) $\neg r \rightarrow s$

hypothesis

(v) s

using modus ponens on
(iv) & (v)

(vi) $s \rightarrow t$

Hypothesis

(vii) t

using modus ponens on (vi) &
(vii).

Show that the hypothesis "If you send me an email message, then I will finish writing the program." "If you do not send me an email message, then I will go to sleep early". "If I go to sleep early then I will wake up feeling refreshed" leads to the conclusion "If I donot finish writing the program, then I will wake up feeling refreshed by using rules of inference.

Solⁿ:

- let p : You send me an email message
- q : I will finish writing the ~~mes~~ program
- r : I will go to sleep early
- s : I will wake up feeling refreshed

Hypothesis

$$\begin{aligned} p &\rightarrow q \\ \neg p &\rightarrow r \\ r &\rightarrow s \\ \cancel{p \rightarrow s} \end{aligned}$$

Conclusion

$$\neg q \rightarrow s$$

steps

Reasons

- | | | |
|-------|-----------------------------|---|
| (i) | $p \rightarrow q$ | Hypothesis |
| (ii) | $\neg q \rightarrow \neg p$ | Contrapositive of (i) |
| (iii) | $\neg p \rightarrow r$ | Hypothesis |
| (iv) | $\neg q \rightarrow r$ | using hypothetical syllogism (ii) & (iii) |
| (v) | $r \rightarrow s$ | Hypothesis |
| (vi) | $\neg q \rightarrow s$ | using hypothetical syllogism in (iv) & (v). |

Resolution principle.

This is another way of proving that an argument is correct.

- Literal: A variable or a negation of variable.
- Sum: A disjunction of literals. (or) / clause
- Product: A conjunction of literals (and)

Negation of conclusion.

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

$$C1: p$$

$$C2: p \rightarrow q \equiv \neg p \vee q$$

$$C3: \neg q$$

$$C4: q$$

$$C5: \square$$

[From C1 & C2
 p & $\neg p$ are complementary of each other so they are cancelled
deleted]

From C3 & C4.

check the validity of the argument by using resolution principle

"If today is tuesday, I have a test in mathematics or economics." "If my economics professor is sick, I will be not have a test in economics." "Today is Tuesday and my economics professor is sick therefore I have a test in mathematics."

let p : Today is tuesday

q : I have a test in mathematics.

r : I have a test in economics

s : Economics professor is sick.

Hypothesis:	$p \rightarrow (q \vee r)$	Conclusion
	$s \rightarrow \neg r$	q
	$p \wedge s$	

putting in clause form:

$$\begin{array}{l} C1: p \rightarrow (q \vee r) \\ : \neg p \vee (q \vee r) \end{array}$$

$C_2: \neg S \vee \neg T$

$C_3: \neg P$

$C_4: S$

$C_5: \neg Q$

C_5 : Resolvent of C_1 and C_2 is obtained as

$C_5: \neg P \vee \neg T$

Resolvent of C_3 and C_5 is obtained as

$C_7: \neg T$

Resolvent of C_4 and C_7

$C_8: T$

Resolvent of C_8 and C_5

$C_9: \square$

As we have reached to empty clause so by using resolution principle, we can say that the given statement is valid.
argument

7) If you do every problem in ^{this book} school then you will learn discrete mathematics

You learn discrete mathematics. Therefore you did every problem in school, this book.

p: You did every problem in this book.
 q: You learned discrete mathematics

$$\frac{p \rightarrow q}{q} \therefore p$$

$$\Rightarrow ((p \rightarrow q) \wedge q) \rightarrow p$$

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$((p \rightarrow q) \wedge q) \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

The expression $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology because it is false when p is false and q is true so the given argument is not a ~~tautology~~ valid argument

- 2) If you do every problem in this book then you will learn discrete mathematics.
 If you didn't do every problem in this book, therefore you didn't learn discrete mathematics.

Solⁿ: let

p: You did every problem in this book
 q: You learned discrete mathematics.

$$\frac{p \rightarrow q}{\neg q} \therefore \neg p$$

$$\Rightarrow ((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$$

p	q	$\neg q$	$\neg p$	$p \rightarrow q$	$p \rightarrow q \wedge \neg p$	$p \rightarrow q \wedge \neg p$
T	T	F	F	T	F	T
T	F	T	F	F	F	T
F	T	F	T	T	T	F
F	F	T	T	T	T	T

The expression $(p \rightarrow q) \wedge \neg p \rightarrow \neg q$ is not a tautology because it is false when p is false and q is true so the given argument is not a valid argument.

Rules of inference for quantified statements.

Rules of inference	Name
(i) $\forall x p(x)$ $\therefore p(c)$ where c is domain of variable x (particular element)	Universal Instantiation
(ii) $p(c)$ for arbitrary c $\therefore \forall x \in p(x)$	Universal generalization
(iii) $\exists x p(x)$ $\therefore p(c)$ for some element c $\therefore \exists x p(x)$	Existential Instantiation
(iv) $p(c)$ for some element c $\therefore \exists x p(x)$	Existential Generalization

hypothesis

- Q Show that the premises, "Everyone in this discrete mathematics class has taken a course in computer science" and "Maria is a student in this class." imply the conclusion "Maria has taken a course in computer science", by using rules of inference.

Sol:-

let $DC(x)$: "x is in the discrete mathematics class"

$CC(x)$: "x has taken a course in computer science"

Hypothesis (Premises)

Conclusion

$\forall x (DC(x) \rightarrow CC(x))$

$CC(Maria)$

$DC(Maria)$

Steps

Reason

(i) $\forall x (DC(x) \rightarrow CC(x))$

(i) Hypothesis

(ii) $DC(Maria) \rightarrow CC(Maria)$

(ii) using Universal instantiation on (i)

(iii) $DC(Maria)$

(iii) Hypothesis

(iv) $CC(Maria)$

(iv) By using Modus ponens on (ii) & (iii).

imp

Show that the premises "A student in this class has not read the book" and "Everyone in this class has passed the first exam" imply the conclusion "someone who passed the 1st exam has not read the book by using rules of inference."

Solⁿ

let $C(x)$: "x is in this class"

$B(x)$: "x has read the book"

$P(x)$: "x has passed the first exam"

Hypothesis

$$\exists x (C(x) \wedge \neg B(x))$$

$$\forall x (C(x) \rightarrow P(x))$$

Conclusion

$$\exists x (P(x) \wedge \neg B(x))$$

Steps

Reasons

(i) $\exists x (C(x) \wedge \neg B(x))$

(i) Hypothesis.

(ii) $C(a) \wedge \neg B(a)$

(ii) Using existential instantiation on (i)

(iii) $C(a)$

(iii) By using simplification rule on (ii)

(iv) $\neg B(a)$

(iv) "

(v) $\forall x (C(x) \rightarrow P(x))$

(v) Hypothesis.

- (vi) $c(a) \rightarrow p(a)$ (vi) using universal instantiation on (v).
- (vii) $p(a)$ (vii) By using modus ponens on (iii) & (vi).
- (viii) $p(a) \wedge \neg b(a)$ (viii) By using conjunction on (iv) and (vii).
- (ix) $\exists x (p(x) \wedge \neg b(x))$ (ix) using Existential Generalization on (viii).

Construct an argument using rules of inference to show that the hypothesis "All movies produced by John Sayles are wonderful" "John Sayles produced a movie about coal miners" imply the conclusion "There is a wonderful movie about coal miners." You are required to show each step before you come to the desired conclusion from the hypothesis.

Sol:-

let $J(x)$: "x is a movie produced by John Sayles"

$w(x)$: "x is a wonderful movie"

$c(x)$: "x is a movie about coal miners."

Hypothesis

$\forall x (J(x) \rightarrow w(x))$
 $\exists x (J(x) \wedge c(x))$

Conclusion

$\exists x (w(x) \wedge c(x))$

Steps

Reasons

- | | | |
|--------|---|--|
| (i) | $\forall x (J(x) \rightarrow W(x))$ | (i) hypothesis. |
| | $J(\text{Cool miner}) \rightarrow W(\text{Cool})$ | |
| (ii) | $J(a) \rightarrow W(a)$ | (ii) Universal instantiation |
| (iii) | $\exists x (J(x) \wedge C(x))$ | (iii) Hypothesis. |
| (iv) | $J(a) \wedge C(a)$ | (iv) using existential instantiation on (iii) |
| (v) | $J(a)$ | (v) By simplification rule on (iv), |
| (vi) | $C(a)$ | (vi) " |
| (vii) | $J(a)$ $W(a)$ | (vii) Using modus ponens on (ii) & (v). |
| (viii) | $W(a) \wedge C(a)$ | (viii) using conjunction on (vi) & (vii). |
| (ix) | $\exists x (W(x) \wedge C(x))$ | (ix) using Existential Generalization on (viii). |

Proof:

Proofs are the valid arguments used to establish the truth of the mathematical statement.

Mainly there are two types of proof.

- Formal proofs
- Informal proofs

Formal proofs: The arguments which are used to show that the statements (involved in) involving proposition and quantifiers statements are true, are formal proofs, where all steps are supplied and the rule for each ~~inform~~ step is given.

Informal proof:

Informal proof can explain to human why theorems are true.

Types of formal proofs

- (i) Direct proof
- (ii) Indirect proof

Direct proof

The direct proof of the conditional statement i.e. $p \rightarrow q$ is given by

- first we assume that p to be true and by using various steps and processes, we show that q must be true.

Prove that if n is an odd integer then n^2 is odd by using direct proof method.

Sol:-

Any integer n is said to be even if there exists an integer ' k ' such that $n = 2k$.

$$\text{odd} \rightarrow n = 2k + 1$$

p : n is odd

q : n^2 is odd

$$p \rightarrow q$$

let p be true i.e. n is odd is true
from the definition of odd integer, we can express n as $n = 2k + 1$ for some integer k .

Now, we have to show that q must be true i.e. n^2 is odd.

Now,

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

which is in the form of $2k + 1$. So q is true i.e. n^2 is odd.

when p holds true, it is found that q is true so the given conditional statement "if n is an odd integer then n^2 is odd" i.e. $p \rightarrow q$ is true by using direct proof

method.

If show by using direct proof method that if m and n are both perfect squares then mn is also a perfect square.

Solⁿ:-

Any integer ' a ' is said to be perfect square if there exists integer ' b ', such that $a = b^2$.

p : m and n are both perfect squares
 q : mn is perfect square.

$p \rightarrow q$

Let p be true, i.e. m and n are both perfect squares, we can express m and n as by defⁿ of perfect square:
 $m = b^2$

Also, $n = a^2$ for some integers a & b

Now we have to show that mn is also a perfect square.

$$mn = b^2 a^2$$

$$mn = (ab)^2$$

which is also true in the form of $a = b^2$ so, q is true i.e. mn is a perfect square.

When p holds true, it is found that q is true so the given conditional statement $p \rightarrow q$ is true by using direct proof method.

Indirect proof:

Proofs of the theorem that don't start with the hypothesis and end with conclusion.

Types of indirect proof:

- (i) proof by contraposition.
- (ii) proof by contradiction.

Proof by contraposition

It uses the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive $\neg q \rightarrow \neg p$.

The proof by contraposition of the conditional statement $p \rightarrow q$ is given by

First, we assume that $\neg q$ to be true and by using various steps and processes we show that $\neg p$ is true.

Q. If n is an integer and $3n+2$ is odd, then n is odd by using Show this by using proof by contraposition.

p : $3n+2$ is odd

q : n is odd

Let $\neg q$ is true; n is not odd which implies n is even. By defⁿ of even integer,

$n = 2K$, for some integer K

Now, we have to show that $\neg p$ to be true

$$\begin{aligned}\text{Now, } 3n+2 &= 3 \times 2K + 2 \\ &= 6K + 2 \\ &= 2(3K+1)\end{aligned}$$

which shows $3n+2$ is even

i.e. $\neg p$ is true

whenever $\neg q$ is true, it is found that $\neg p$ is true so the given original conditional statement $p \rightarrow q$ is true

Because the negation of the conclusion of the conditional statement implies that the hypothesis is false therefore the conditional statement is true.

Prove that if $n = ab$ where a & b are the integers then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ by using proof by contraposition.

let p denote $p : n = ab$.

$q : a \leq \sqrt{n} \vee b \leq \sqrt{n}$

let $\neg q$ is true i.e. $\neg q = \neg (a \leq \sqrt{n} \vee b \leq \sqrt{n})$

$$= \neg (a \leq \sqrt{n}) \wedge \neg (b \leq \sqrt{n})$$

$$= (a > \sqrt{n}) \wedge (b > \sqrt{n})$$

$$\Rightarrow a > \sqrt{n} \text{ and } b > \sqrt{n}$$

Now we have that $\neg p$ to be true

Now, multiplying $a > \sqrt{n}$ and $b > \sqrt{n}$, we

$$\begin{aligned} ab &> n \\ \Rightarrow ab &\neq n \end{aligned}$$

Because, the negation of the conditional conclusion of the conditional statement is $\neg q$ implies that the hypothesis is false i.e. $\neg p$ is false $\Rightarrow p$ is true so the given conditional statement $p \rightarrow q$ is true

(ii) proof by contradiction

→ proof by contradiction for propositional statement.

→ proof by contradiction for conditional statement.

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction

Solⁿ:

→ A real number 'r' is said to be rational if r can be expressed in the form $r = \frac{p}{q}$ where $q \neq 0$ and p and q do not have common factors.

let p denote " $\sqrt{2}$ is irrational"

let $\neg p$ be true i.e. $\sqrt{2}$ is rational.

Since $\sqrt{2}$ is rational by the definition of rational number,

$$\sqrt{2} = \frac{a}{b}$$

$$\text{or } \sqrt{2} b = a$$

Squaring on both sides,

$$\text{or } 2b^2 = a^2$$

$$\text{or } a^2 = 2b^2.$$

Here a^2 is even as it is in the form $a^2 = 2b^2$

Since a^2 is even, a is even so a can be expressed as $a = 2K$, for some integer K .

Now, substituting value of $a = 2K$ in eqⁿ $2b^2 = a^2$ we get,

$$2b^2 = (2K)^2$$

$$2b^2 = 4K^2$$

$$b^2 = 2K^2$$

From above we can conclude that b^2 is even so b is also even. So, b can be expressed as $b = 2L$ for some integer L .

Here, a and b both have common factors which contradicts that $\sqrt{2}$ is rational. So our assumption " $\neg p$ is true" is false i.e. so the given original statement p is true i.e. $\sqrt{2}$ is irrational.

Proof by contradiction for conditional statements:

The proof by contradiction of any conditional statement $p \rightarrow q$ is given by:

First, we assume that p and $\neg q$ to be true.

Finally, we show that $\neg p$ is true which contradicts our assumption that p is true.

eg:

Give a proof by contradiction to the theorem: "If $3n+2$ is odd then n is odd."

Solⁿ: let $p: 3n+2$ is odd

$q: n$ is odd

let p and $\neg q$ be true i.e. $3n+2$ is odd and n is not odd then n is odd.

so

let $p: 3n+2$ is odd

$\neg q: n$ is odd

let p and $\neg q$ be true i.e. $3n+2$ is odd and n is not odd, which implies n is even.

From the definition of even integer,

$n = 2k$ for some integer k

Now,

$$3n+2 = 2(2k)+2$$

$$= 6K+2$$

$= 2(3K+1)$, which is even i.e. $\neg p$ is true.

Here, $\neg p$ is true which contradicts our assumption that p is true so our assumption is wrong. So, the given original conditional statement is true.

Vacuous Proof & Trivial Proof of Conditional Statement

$p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

For vacuous proof.

The vacuous proof of any conditional statement is given by:

We show that p part to be false no matter whatever the truth value of q is. So, finally, we can say that $p \rightarrow q$ is true by vacuous proof.

$p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

for trivial proof

The trivial proof of any conditional statement is given by:

We show that q part to be true no matter

whatever the truth value of p is. So, finally we can say that $p \rightarrow q$ is true by trivial proof.

eg 1) Show that the proposition $p(0)$ is true where p is "If n is greater than 1 then n^2 is greater than n ." and the domain consists of all integers by giving a vacuous proof.

Sol:- The $p(0)$ is obtained by substituting $n=0$. Here "0 is greater than 1, which is false. So $p(0)$ is true by vacuous proof.

2) Let $p(n)$ be, "If $a \times b$ are positive integers with $a \geq b$ then a^n is greater or equal to b^n " (a^n), where domain consists of all integers. Show that $p(0)$ is true by trivial proof method.

Ans: $p(0)$ can be obtained by substituting $n=0$. Here, $a^0 \geq b^0$ i.e. $1 \geq 1$ which shows that $p(0)$ is true by trivial proof.

Proof by Equivalence

The proof by equivalence of any bi-conditional statement $p \leftrightarrow q$ is given by

- i) First, we show that $p \rightarrow q$ is true and secondly we show that $q \rightarrow p$ is true.

Example:

Prove the theorem "If n is a positive integer then n is odd if and only if n^2 is odd."

Solⁿ: let p denote $p: n$ is odd
 $q: n^2$ is odd.

Now, first we show that $p \rightarrow q$ is true by direct proof method.

let p be true i.e. n is odd
then by definition of odd integer

$$n = 2K + 1 \text{ for some integer } K.$$

Now squaring on both sides, we get

$$n^2 = (2K + 1)^2$$

$$n^2 = 4K^2 + 4K + 1$$

$$= 4K(K + 1)$$

$$= 2K \cdot 2$$

$$= 2(K^2 + 2K + 1)$$

$$= 2(K + 1)^2$$

$$= 2(2k^2 + 2k) + 1$$

$$= 2(2k^2 + 2k) + 1 \quad \text{which is in the form of}$$

$$2k+1 \quad \text{so } q \text{ is true i.e. } n^2 \text{ is odd}$$

So, the given conditional statement $p \rightarrow q$ is true

Now, we have to show that $q \rightarrow p$ is true. by giving we show that a proof by contraposition (Now, By contrapositive, $\neg p \rightarrow \neg q$ is true) - a positive

Now, let $\neg p$ be true i.e. n is not odd which implies n is even.

Then,

from the definition of even integers,

$$n = 2k$$

squaring on both sides

$$n^2 = 4k^2$$

$$n^2 = 2 \cdot 2k^2 \quad \text{which is in the form of } 2k$$

i.e. n^2 is even, so, $\neg q$ is true i.e. n^2 is even.

So, the given conditional statement $q \rightarrow p$ is true by proof by contraposition.

Finally, the given conditional statement "If n is a positive integer then n is odd if and only if n^2 is odd" is true by proof by

equivalence.

Show that these statements about integer n are equivalent.

p_1 : n is even.

p_2 : $n-1$ is odd

p_3 : n^2 is even

Solⁿ:

First, we show that $p_1 \rightarrow p_2$ is true by direct proof method.

Let p_1 be true i.e. n is even.
Then by definition of even integer.

$$n = 2K$$

Now,

$$n-1 = 2K-1$$

$$= 2K-2+1$$

$$n-1 = 2(K-1)+1$$

i.e. p_2 is true i.e. $n-1$ is odd.

So, the given conditional statement $p_1 \rightarrow p_2$ is true by direct proof method.

Now,

we show that $p_2 \rightarrow p_3$ is true by direct proof method.

let p_2 be true, i.e. $n-1$ is odd.
then by defⁿ of odd integer

$$n-1 = 2K+1 \quad \text{for some integer } K$$

~~Square~~
Squaring on both side,

$$n = 2K+2$$

$$n^2 = (2K+2)^2$$

$$n^2 = 4K^2 + 8K + 4.$$

$$n^2 = 2(2K^2 + 4K + 2)$$

$$n^2 = 2(2K^2 + 4K + 2)$$

which is in the form of $2K$ i.e.
 p_3 is true, i.e. n^2 is even

So, we can conclude that $p_2 \rightarrow p_3$ is
true by direct proof method.

Again,

we show that $p_3 \rightarrow p_1$ is true by
~~direct proof~~ ^{by contraposition.} method. $\neg p_1 \rightarrow \neg p_3$

let $\neg p_3$ be true i.e. n^2 is even.
Now, by defⁿ of even n

let $\neg p_1$ be true. i.e. n is odd.

then, By defⁿ of odd integer, we can write
 $n = 2K+1$ for some integer K .

squaring on both sides,

$$n^2 = (2K+1)^2$$

$$n^2 = 4K^2 + 4K + 1$$

$$n^2 = 2(2K^2 + 2K) + 1$$

i.e. n^2 is odd so $7p_3$ is true.

∴ So $7p_1 \rightarrow 7p_3$ is true which means $p_3 \rightarrow p_1$ is true.

Finally, the given statements are equivalent.

Proof by Mathematical Induction.

(i) Basis step

→ we show that $p(1)$ and $p(2)$ ^{to} be true of the mathematical statement $p(n)$.

(ii) Inductive step

→ we assume that $p(K)$ holds true for arbitrary integer K and we show that $p(K+1)$ holds true.

Show that $1+2+3+\dots+n = \frac{n(n+1)}{2}$ by

mathematical induction method where n is a positive integer. by using mathematical

Solⁿ:-

let $p(n)$ denote $1+2+\dots+n = \frac{n(n+1)}{2}$

Basis step:-

for $n=1$,

$$p(1) \Rightarrow 1 = \frac{1(1+1)}{2}$$

$$\Rightarrow 1 = 1 \text{ which is true}$$

$\therefore p(1)$ holds true

for $n=2$,

$$p(2) \Rightarrow 1+2 = \frac{2(2+1)}{2}$$

$$\Rightarrow 3 = 3$$

which is true

$\therefore p(2)$ holds true.

let $p(k)$ holds true for arbitrary integer k i.e.

$$1+2+\dots+k = \frac{k(k+1)}{2} \text{ is true.}$$

Now, we have to show that $p(k+1)$ holds true i.e.

$$1+2+\dots+k+1 = \frac{k+1}{2}(k+2)$$

$$\text{L.H.S} = 1+2+\dots+k+k+1 = \frac{k+1}{2}(k+2)$$

&

$$\text{O.S.} = \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1)+2k+2}{2}$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \text{R.H.S.}$$

Here, $p(k) \Rightarrow p(k+1)$ holds true whenever $p(k)$ is true. So, the given mathematical statement $p(n)$ is true by mathematical induction.

Prove that $n < 2^n$ for all +ve integer n by using mathematical induction.

Solⁿ:-

Let $p(n)$ denote $n < 2^n$

Basis step:

for $n=1$, $p(1) \Rightarrow 1 < 2^1$ which is true.

$\therefore p(1)$ holds true

for $n=2$

$$p(2) \Rightarrow 2 < 2^2$$

$2 < 4$ which is true

$\therefore p(2)$ holds true

Inductive step:

let $p(k)$ holds true i.e. $k < 2^k$ is true

Now, we have to show that

$p(k+1)$ holds true

i.e. $k+1 < 2^{k+1}$ is true

$$k < 2^k \quad \dots \quad \textcircled{i}$$

$1 < 2^k$ is always true for positive integer
— $\textcircled{ii}$.

Adding $\textcircled{i}$ & $\textcircled{ii}$, we get

$$\Rightarrow k+1 < 2^k + 2^k$$

$$\Rightarrow k+1 < 2 \cdot 2^k$$

$$\Rightarrow k+1 < 2^{k+1}$$

Here $P(K+1)$ holds true whenever $P(K)$ is true.
So, the given mathematical statement $P(n)$ is true by ^{proof by} mathematical induction.

Show that $1+2+2^2+\dots+2^n = 2^{n+1}-1$ for all non-negative integers, by mathematical induction.

Basis Step:

Let $P(n)$ denote $1+2+2^2+\dots+2^n = 2^{n+1}-1$.

$$\text{When } n=0, \quad 1 = 2^1 - 1$$

$$1 = 1$$

$$\text{L.H.S} = \text{R.H.S}$$

$$\text{When } n=1, \quad 1+2 = 2^{1+1} - 1$$
$$3 = 3$$

$$\text{L.H.S} = \text{R.H.S.}$$

So, $P(0)$ & $P(1)$ holds true.

Inductive step:

Let $P(K)$ holds true i.e.

$$1+2+2^2+\dots+2^K = 2^{K+1}-1 \text{ is true.}$$

Now, we have to show that $P(K+1)$ holds true

$$\text{i.e. } 1+2+2^2+2^3+\dots+2^{K+1} = 2^{K+2}-1.$$

$$\text{L.H.S} = 1+2+2^2+\dots+2^{K+1}$$

$$= 1 + 2 + 2^2 + 2^3 + \dots + 2^k + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^{k+1}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

$$= R.H.S$$

which holds true for $P(k+1)$ when $P(k)$ is true.

Hence, by mathematical induction, $P(n)$ is true.

$$1 + 2 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 1.$$

Hence Proved #

Tableaux Method

- Tableaux Method in a propositional logic is the method in which set of rules are applied symmetrically in a formula or a set of formulae to check if it is consistent (satisfiable) or inconsistent (unsatisfiable).
- The main principle is to break down the formula or set of formulae into its component upto smaller & ones until the complementary pair of literals is obtained or no further expansion is possible.

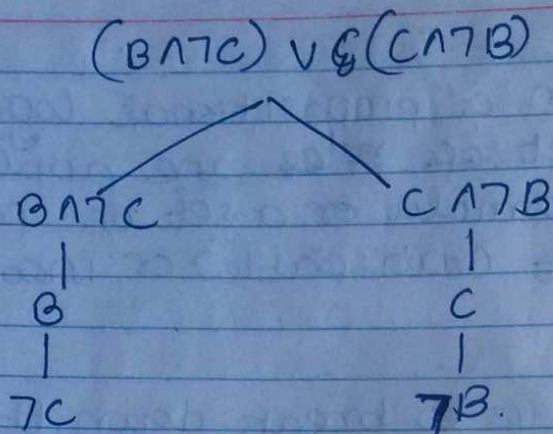
Tableaux Derivation Rules.

$$\textcircled{i} \quad \begin{array}{c} B \wedge C \\ | \\ B \\ | \\ C \end{array}$$

$$\textcircled{ii} \quad \begin{array}{c} \neg(B \wedge C) \\ = \neg B \vee \neg C \end{array} \quad \textcircled{iii} \quad \begin{array}{c} \neg(B \vee C) \\ = \neg B \wedge \neg C \\ \begin{array}{cc} \neg B & \neg C \end{array} \end{array}$$

$$\textcircled{iv} \quad \begin{array}{c} B \rightarrow C = \neg B \vee C \\ \begin{array}{cc} \neg B & C \end{array} \end{array}$$

$$\begin{aligned} \textcircled{v} \quad & \neg(C \leftrightarrow C) \\ & \neg[(C \rightarrow C) \wedge (C \rightarrow C)] \\ & \equiv \neg(C \rightarrow C) \vee \neg(C \rightarrow C) \\ & \equiv \neg C \vee \neg C \\ & \equiv (C \wedge \neg C) \vee (C \wedge \neg C) \end{aligned}$$



Steps to draw tableau of any set of formulae

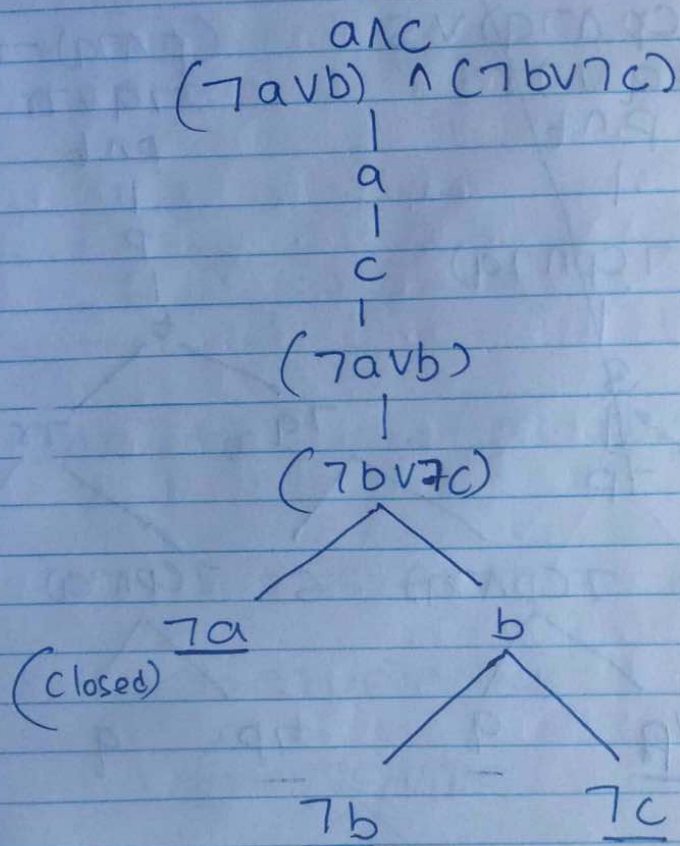
- Step I : List the set of formulae in columnar form
- Step II : If $\phi(x)$ is not the literal, apply the appropriate rule
- Step III : After each step, we check for satisfiability along branches.
- Step IV : If a literal p and $\neg p$ appear on the same path of the tableau, then the branch is not satisfiable i.e. inconsistency is maintained and such path is said to be contradictory or closed path (finished)
- Step V : Even if one path remains non-contradictory or open then the formula ϕ at the root of a tableau is consistent.

Example:

$$\phi = \{ a \wedge c, (\neg a \vee b) \wedge (\neg b \vee \neg c) \}$$

- 1) draw the tableau of the above set of formulae and check the ~~so~~ unsatisfiability or satisfiability.

Here, $\phi = \{ a \wedge c, (\neg a \vee b) \wedge (\neg b \vee \neg c) \}$



So, the given set of formulae is unsatisfiable as all the branches are closed.

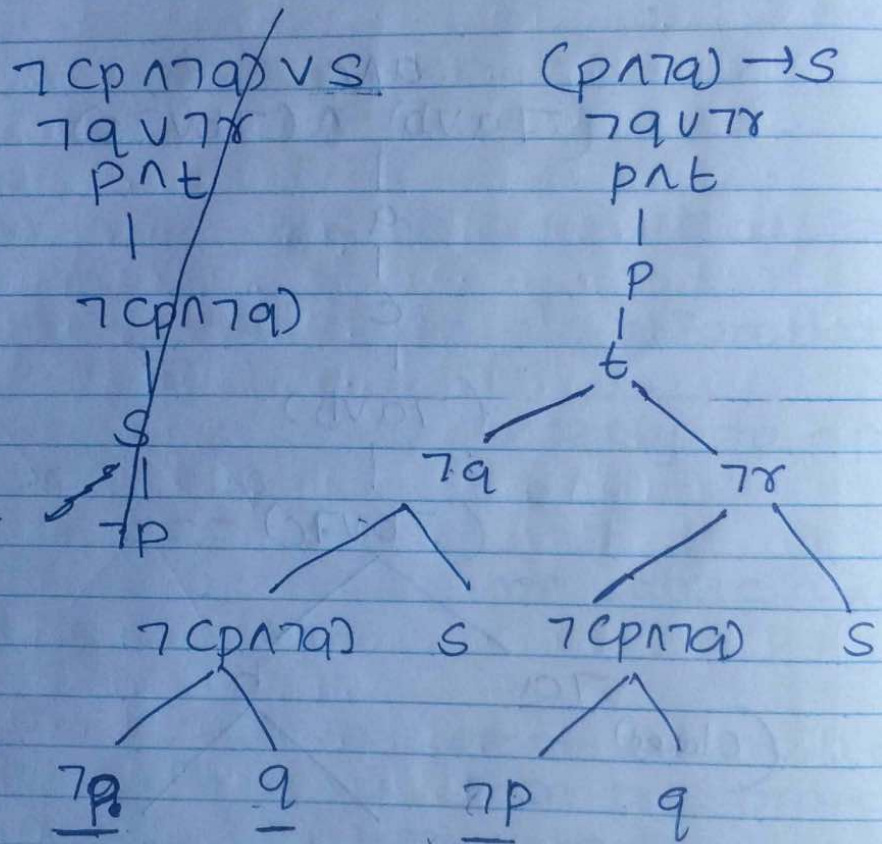
2) $\phi = \{ (p \wedge \neg q) \rightarrow s, \neg q \vee \neg r, p \wedge t \}$

$$\neg p \vee q \text{ vs.}$$

$$(p \wedge q) \rightarrow s = \neg (p \wedge \neg q) \vee s =$$

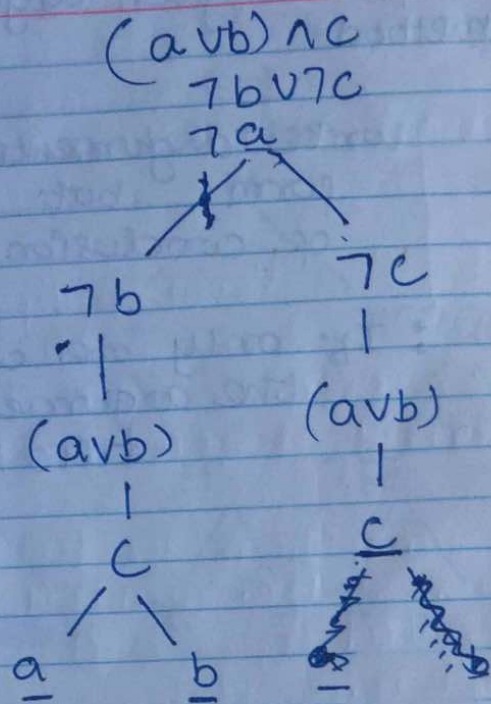
$$\neg q \vee \neg p \vee s = \neg q \vee \neg p \vee s$$

$$p \wedge t = p \wedge t$$

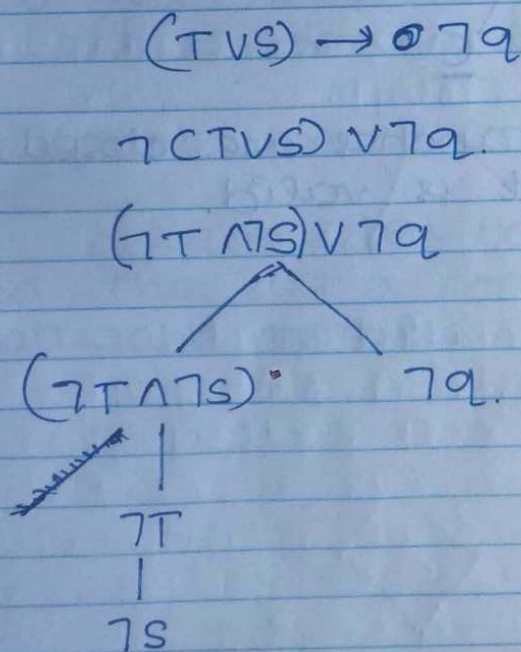


The given set of formulae is ~~unsatisfiable~~ satisfiable.

$$\phi = \{ (a \vee b) \wedge c, \neg b \vee \neg c, \neg a \}$$



Hence, the given set of formulae is unsatisfiable



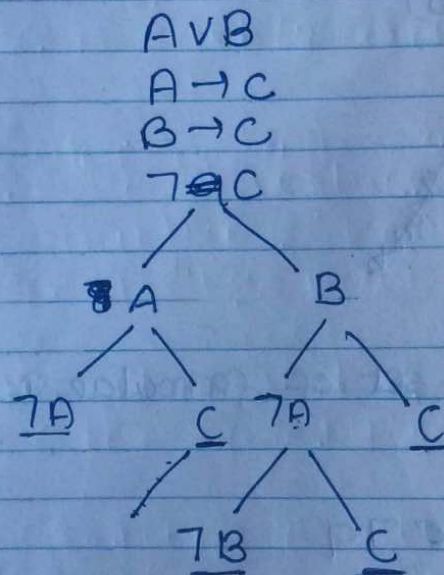
Satisfiable

check the validity of the given argument by using tableau method.

a)
$$\begin{array}{l} A \vee B \\ A \rightarrow C \\ \underline{B \rightarrow C} \\ \therefore C \end{array}$$

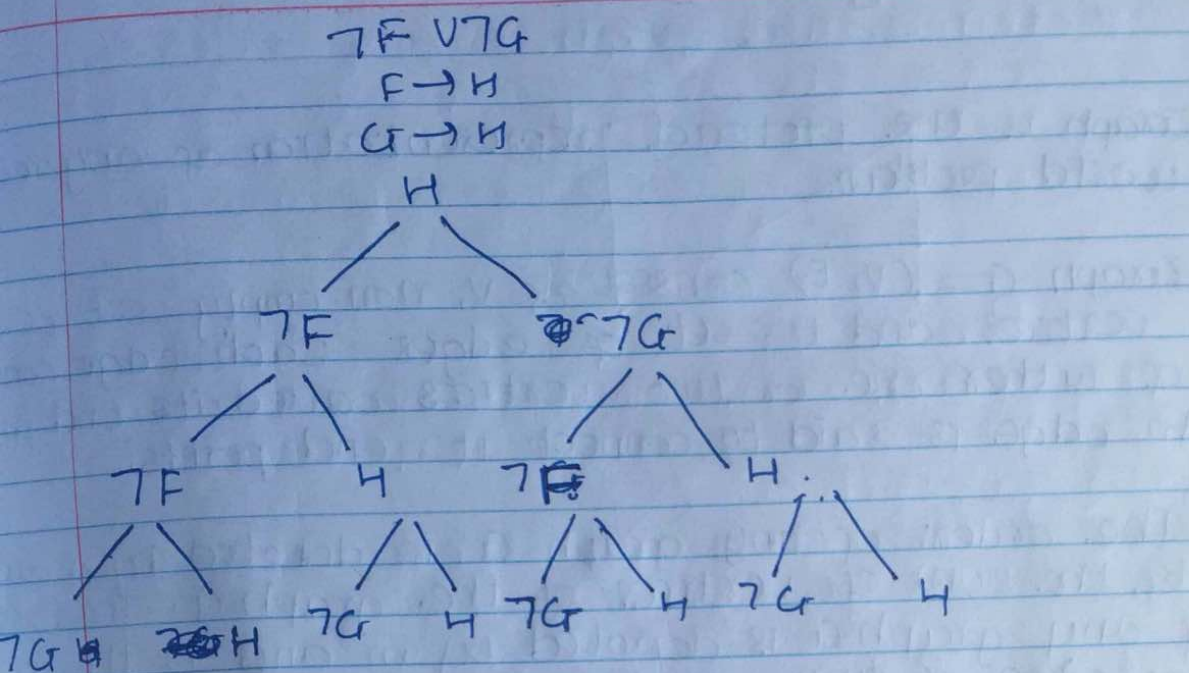
write arguments in columnar form but take negation of conclusion.

: If only one closed also the argument is valid.



Here, all the branches are closed, so the given argument is valid.

b)
$$\begin{array}{l} \neg F \vee \neg G \\ F \rightarrow H \\ \underline{G \rightarrow H} \\ \therefore \neg H \end{array}$$



Here, all the branches are open, so the given argument is open. Invalid.

If a completed tableau for a formula ϕ is open then such a formula ϕ is said to be satisfiable. Give Eg.

Unsatisfiable formula:

If a completed tableau for a formula ϕ is closed, then such a formula ϕ is said to be unsatisfiable formula.

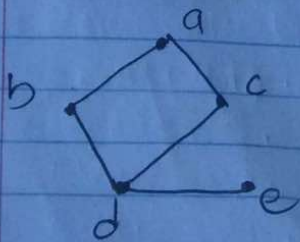
Chapter 4.

Graph Theory (32 marks)

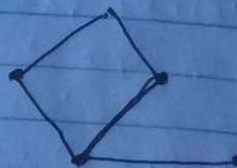
- Graph is the pictorial representation of any real world problem.
- Graph $G = (V, E)$ consist of V , non-empty set of vertices and E , set of edges. Each edge consist of either one or two vertices called its end points. An edge is said to connect its end points.
- The order of any graph G is denoted by n and is the sum of vertices of the graph G . The size of any graph G is denoted by m and is the no. of edge of any graph G .
- Since the set of vertices of any graph is non-empty so the order of any graph is at least one.
- Graph consisting exactly one vertex and no. of edge is called Trivial Graph.

Types of Graph.

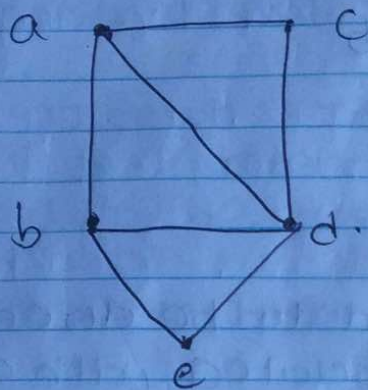
- 1) Labelled and unlabelled graph
- 2) Undirected simple and Multigraph
- 3) Directed simple and Multigraph.



Labelled Graph



Unlabelled graph

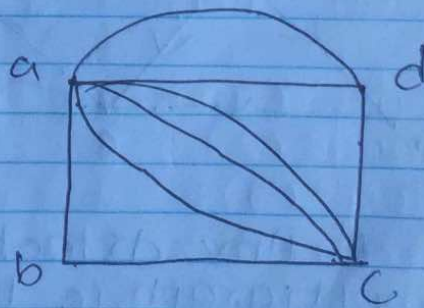
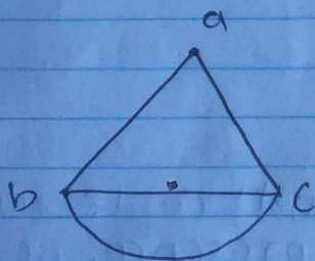


Any undirected graph is said to be simple if there is no presence of parallel edges between two different pair of vertices.

$$V(G) \text{ or } V = \{a, b, c, d, e\}$$

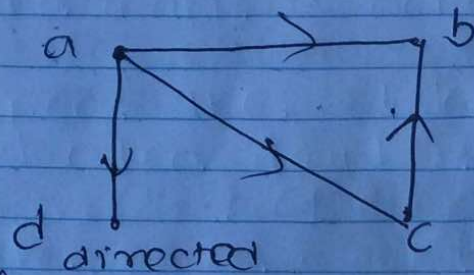
$$E = \{(a, b), (a, c), (a, d), (b, d), (c, d), (d, e), (c, d)\}$$

There is no presence of loop and parallel edges in simple graph.



Any undirected graph G is said to be multigraph if there is presence of multiple edges connecting two vertices. In such graph, there is no presence of loop.

Directed simple graph.



Any graph G is said to be directed simple if each edge is associated with an ordered pair of vertices.

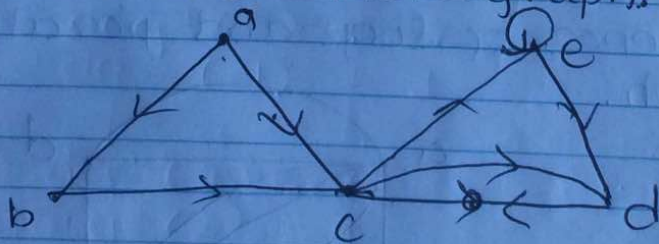
In such graph, there is no presence of parallel edges and loop.

$$V(G) \text{ or } V = \{a, b, c, d\}$$

$$E = \{(a, b), (c, b), (a, c), (a, d)\}$$

Directed simple and multigraph.

Directed multigraph:



Any directed graph G is said to be multigraph if there is presence of parallel edges between two pair of vertices. In such graph, loop is allowed.

$$V = \{a, b, c, d, e\}$$

$$E = \{(a, b), (b, c), (a, c), (c, e), (c, d), (d, c), (e, d)\}$$

A Graph Theory

Terminologies Related to undirected graph:

- Pseudograph: In any undirected graph, if there is presence of loop, then it is said to be pseudograph.

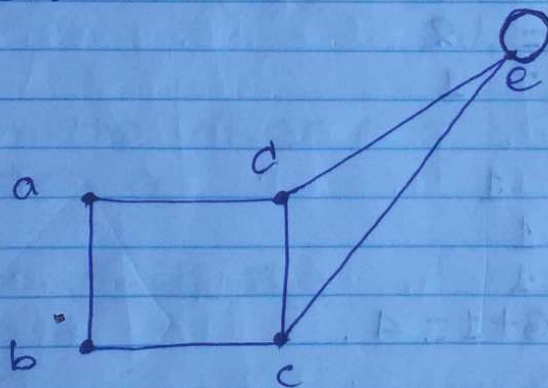
- Adjacent (Neighbours)

$u \Delta v$

Two vertices are said to be adjacent if they are connected by an edge.

- Degree of vertex:

In an undirected graph, the degree of any vertex v , denoted by $\deg(v)$ is the total no. of edges incident with these vertices.



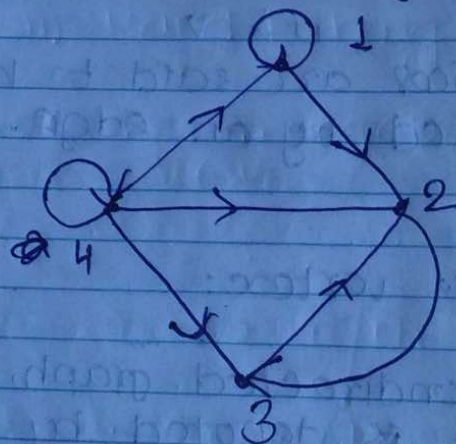
Let the degree of each vertices in the given graph.

Degree of a	$\deg(a) = 2$	$\deg(d) = 3$
	$\deg(b) = 2$	$\deg(e) = 2 + 2$
	$\deg(c) = 3$	$= 4$

Indegree: The indegree of any vertex v in a directed graph is denoted by $\deg^-(v)$

Outdegree: The outdegree of any vertex v in a directed graph is denoted by $\deg^+(v)$

List the indegree & outdegree of each vertices in the given directed graph.



incoming edges : indgree
outgoing edges : outdeg

$$\deg^-(1) = 1 + 1 = 2$$

$$\deg^-(2) = 3$$

$$\deg^-(3) = 2$$

$$\deg^-(4) = 1$$

$$\deg^+(1) = 1$$

$$\deg^+(2) = 1$$

$$\deg^+(3) = 1$$

$$\deg^+(4) = 3 + 1 = 4$$

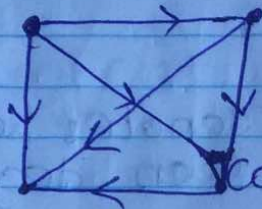
In a round-robin tournament, the Tiger beat the Blue Jays, the Tiger beats the Cardinals, the Tiger beat the Orioles. The Blue Jays beat the Orioles and the Cardinals beat the Orioles. Model this outcome.

with a directed graph.

Solⁿ: Let each team be denoted by vertices.

Tigers

Blue Jays



Orioles

Cardinals

Imp

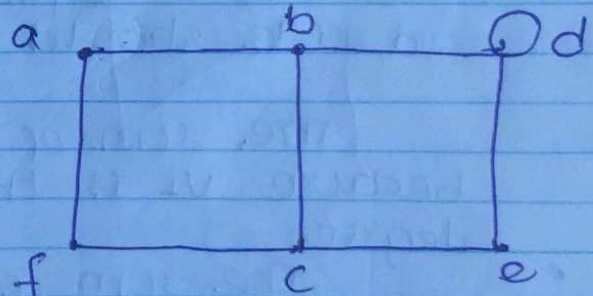
Theorem :

The Handshaking Theorem:

Let $G = (V, E)$ be an undirected graph with e edges then

$$2e = \sum_{v \in V} \deg(v)$$

Here, in the given graph G



total no. of edges
= ~~16~~ 8

$$\text{So, } 2e = 2 \times 8 = 16$$

$$\deg(a) = 2$$

$$\deg(d) = 4$$

$$\deg(b) = 3$$

$$\deg(e) = 2$$

$$\deg(c) = 3$$

$$\deg(f) = 2$$

$$\text{So, } 2e = \sum_{v \in V} \deg(v) = 16, \text{ is even.}$$

2nd

- 2) state and prove the theorem an undirected graph has an even no. of vertices of odd degree.

Proof:

let V_1 and V_2 denotes set of vertices of odd degree and even degree respectively.

then, from Handshaking theorem,

$$2e = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

Proof:-

(let V_1 and V_2 be the set of)

$$1+3+5 = 9$$

$$1+3+5+7 = 16$$

From above equality, we can see that, the left hand side is even, in order to be above equality true, the right hand side should also be even.

The sum of $\deg(v)$ for $v \in V_1$ is even because V_1 is the set of vertices of even degree.

The sum of $\deg(v)$ for $v \in V_2$ should be even since V_2 is the set of vertices of odd degree, the no. of such vertices should be even in order to be sum even.

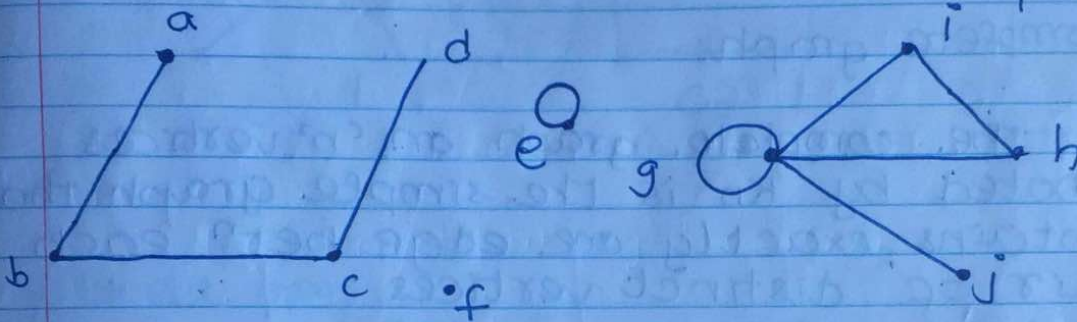
Hence we can say that undirected graph has an even no. of vertices of odd degree.

Theorem:

→ In an directed (digraph) the sum of indegrees = sum of out degrees = no. of edges

$$\text{i.e. } \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

isolated $\deg = 0$
pendent $\deg = 1$



Q) Determine which vertices in the given graph are isolated and pendent. What is the maximum degree? What type of graph is it? Determine the no. of edges and sum of the degrees of its vertices.

Solⁿ:

f vertex is isolated
a, d & j are pendent.

The maximum degree is 5 of vertex g.

The given graph is pseudograph

No. of edges = 9

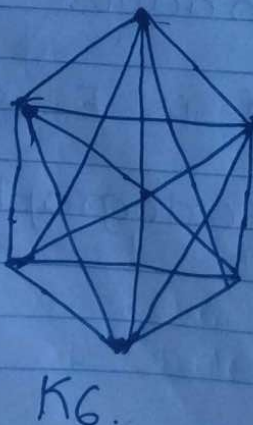
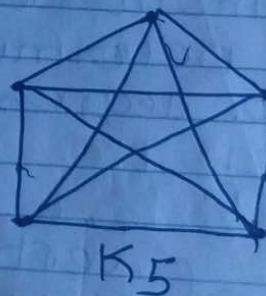
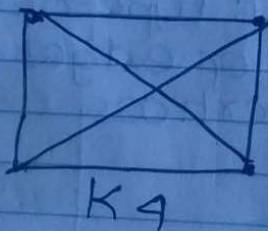
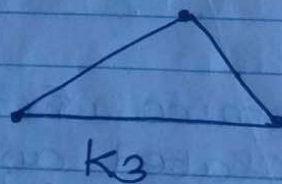
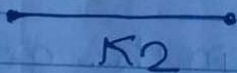
Sum of degree = 18

Different classes of simple graph:

- i) Complete graph
- ii) Cycle graph
- iii) Wheel graph
- iv) n -cube (n -dimensional hypercube)
- v) Bipartite graph
- vi) Complete bi-partite graph.

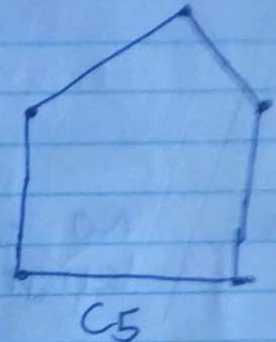
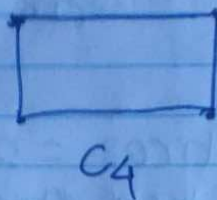
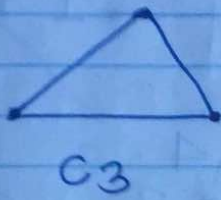
i) Complete graph:

The complete graph on ' n ' vertices denoted by K_n is the simple graph that contains exactly one edge betⁿ each pair of distinct vertices



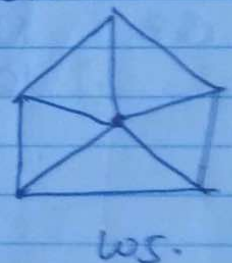
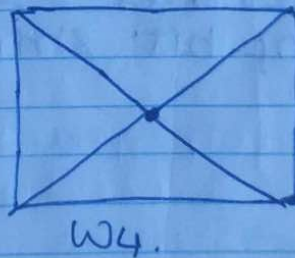
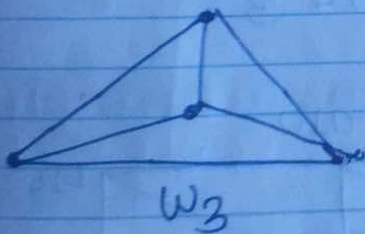
i) Cycle graph:

The cycle graph C_n , for $n \geq 3$ $V_1, V_2, V_3, \dots, V_n$ and edges $(V_1, V_2), (V_2, V_3), \dots, (V_{n-1}, V_n), (V_n, V_1)$



iii) Wheel graph.

When a new vertex is added to a cycle graph C_n and this new vertex is connected to each of the n vertices in C_n then we obtain a wheel graph W_n .

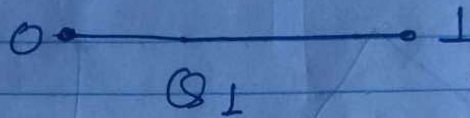


iv) n-cube (n-dimensional hypercube)

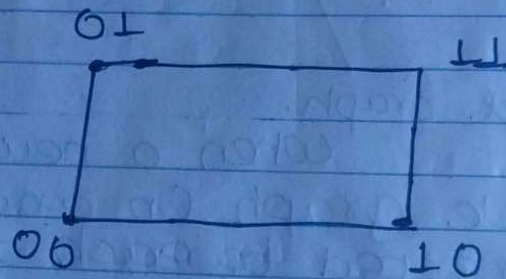
The n -dimensional hypercube, or n -cube denoted by Q_n is the graph that has vertices representing the 2^n bit string of length n . Two vertices are adjacent if and only if

(the bit string) the bit strings that differ in exactly one bit position

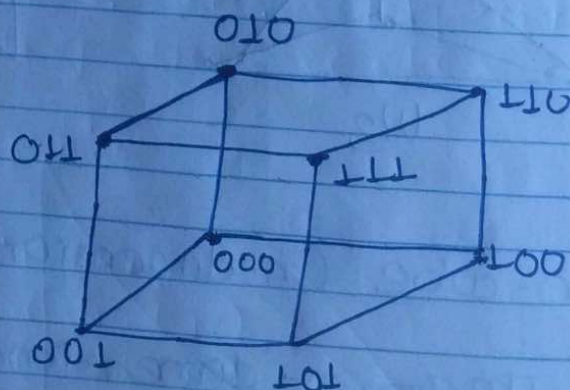
Q1 No. of vertices = $2^n = 2^1 = 2$
length of bit string = $n = 1$



Q2 No. of vertices = $2^n = 2^2 = 4$
length of bit string = $n = 2$

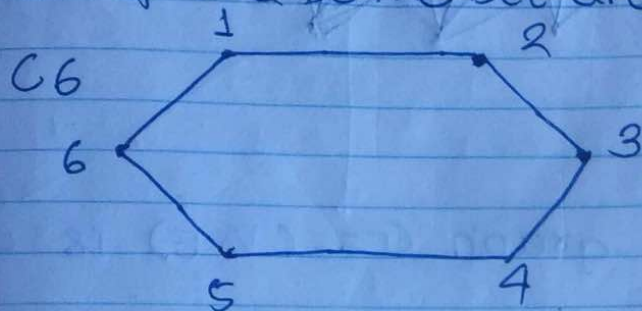


Q3 No. of vertices = $2^3 = 8$
length of bit string = $n = 3$



V) Bipartite Graph
A simple graph G is Bipartite if its vertex set

can be partitioned into two disjoint subsets V_1 & V_2 such that every edge in the graph connects the vertex from the set V_1 to the vertex of set V_2 and no two vertices of the same set are connected by an edge.

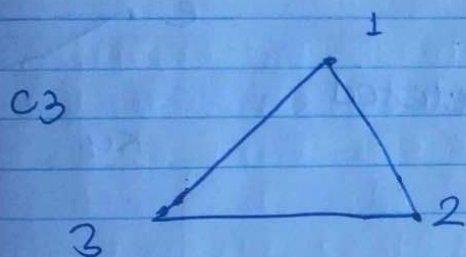
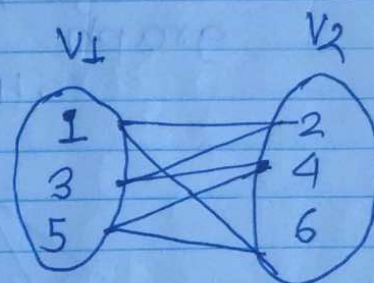


adjacent via neline

The given graph is bipartite

$$V_1 = \{1, 3, 5\}$$

$$V_2 = \{2, 4, 6\}$$



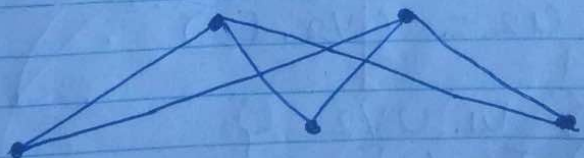
The given graph is not bipartite

ii) Complete Bi-partite graph

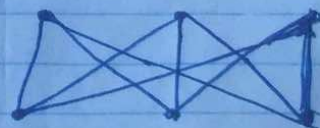
→ denoted by $K_{m,n}$, where $m \leq n$.

$K_{2,3}$, $K_{3,3}$, & $K_{5,5}$

$K_{2,3}$

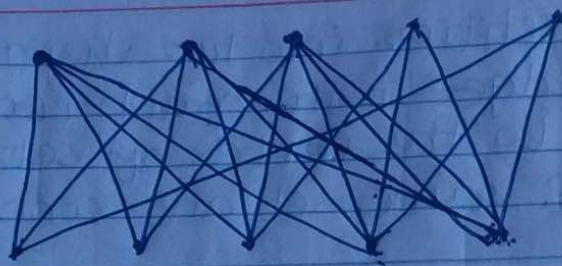


$K_{3,3}$



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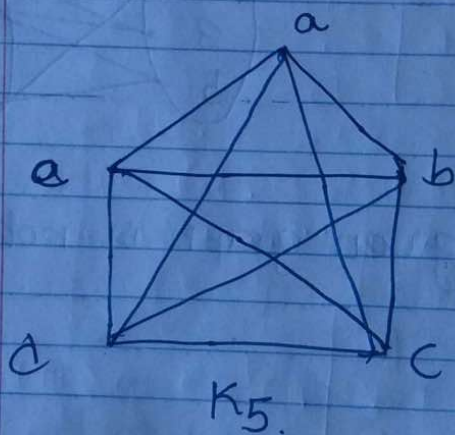
$K_{5,5}$



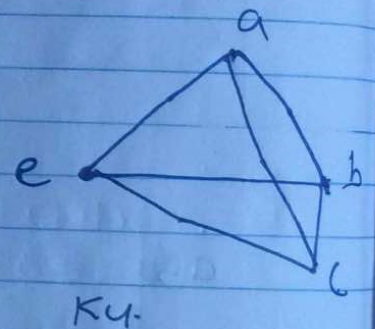
Subgraph

→ A subgraph of a graph $G = (V, E)$ is a graph

$H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$.



a is deleted

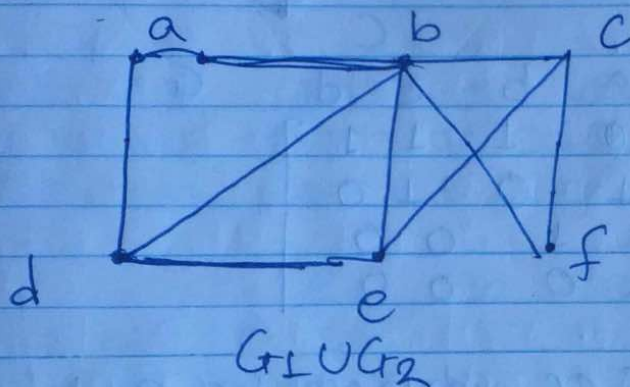
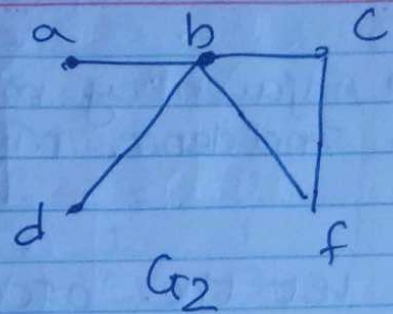
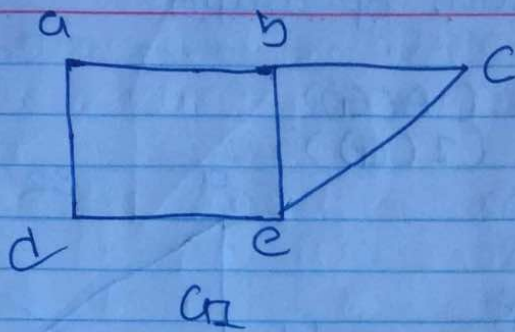


The given graph is the subgraph obtained by deleting the vertex a .

Union of two graphs.

denoted by $G_1 \cup G_2$
where $G_1 = (V_1, E_1)$
 $G_2 = (V_2, E_2)$

$$G_1 \cup G_2 = \left\{ \begin{array}{l} V_1 \cup V_2 \\ E_1 \cup E_2 \end{array} \right\}$$



Representing graphs:

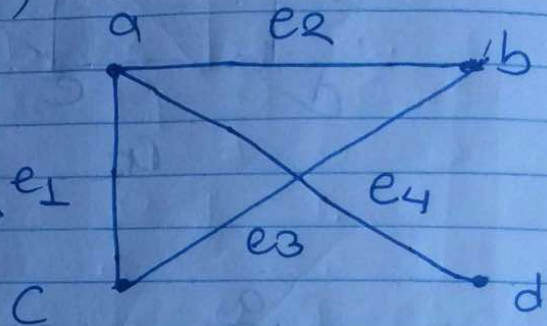
→ Adjacency lists.

vertex	Adjacent vertices
a	b, d, e
b	a, c
c	b, d
d	a, c
e	a

Matrix representation of graph:

- i) → Adjacency matrix ($A(G)$)
- ii) → Incidence matrix ($I(G)$)

Let the order of vertices be a, b, c, d .

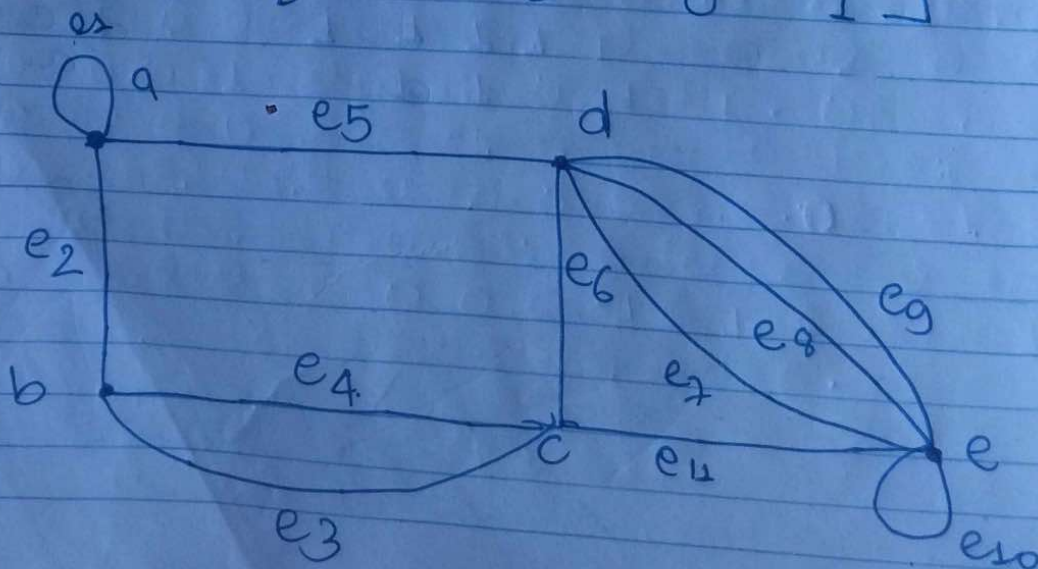


i)

$$A(G) = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Let the order of vertices be a, b, c, d and that of edges be e_1, e_2, e_3, e_4 .

$$I(G) = \begin{matrix} & \begin{matrix} e_1 & e_2 & e_3 & e_4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$



Adjacency matrix.

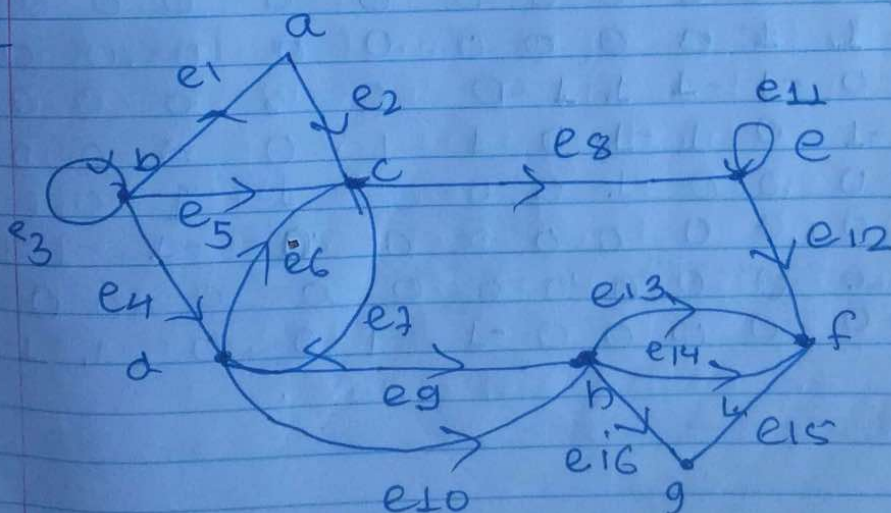
let the order of vertices be a, b, c, d, e

$$A(G) = \begin{matrix} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix} \end{matrix}$$

let the order of vertices be a, b, c, d, e and that of edges be $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}$.

$$\begin{matrix} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

II



let the order of matrices be a, b, c, d, e, f, g, h. so, adjacency matrix is

$$A(G) = \begin{matrix} & \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \end{bmatrix} \end{matrix}$$

let the order of matrix be a, b, c, d, e, f, g, h and the edges be $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$.

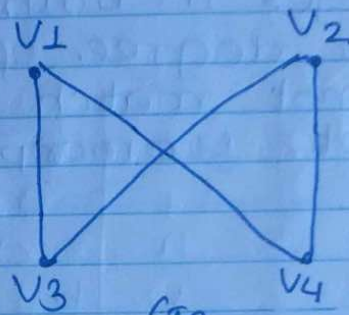
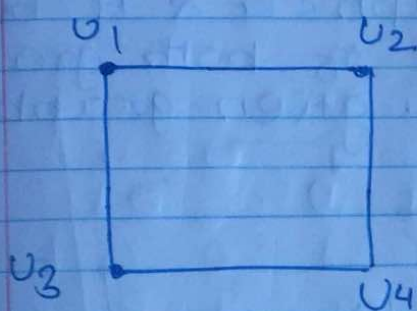
$\Rightarrow$ outgoing = 1

$\Rightarrow$ incoming = -1

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}
a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
b	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0
c	0	-1	0	0	-1	-1	1	1	0	0	0	0	0	0	0	0
d	0	0	0	-1	0	1	-1	0	1	1	0	0	0	0	0	0
e	0	0	0	0	0	0	-1	0	0	0	1	1	0	0	0	0
f	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0
g	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	1	0	0
h	0	0	0	0	0	0	0	0	-1	-1	0	0	1	1	0	1

Graph Isomorphism

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there is a one-to-one and onto function f from V_1 to V_2 with the property that a & b are adjacent in G_1 if & only if $f(a)$ & $f(b)$ are adjacent in G_2 for all a & b in G_1 . Such a function f is said to be isomorphism.



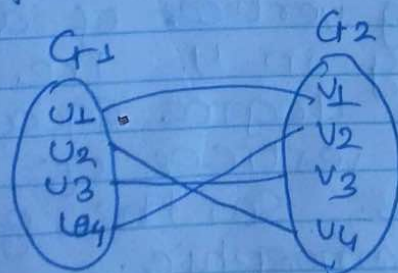
let us define a function f such that

$$f(u_1) = v_1$$

$$f(u_2) = v_4$$

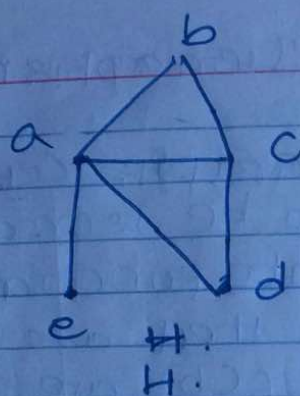
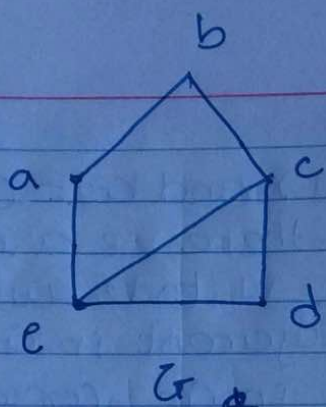
$$f(u_3) = v_3$$

$$f(u_4) = v_2$$

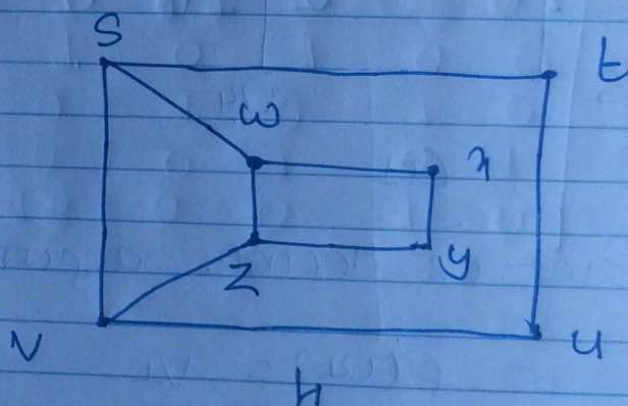
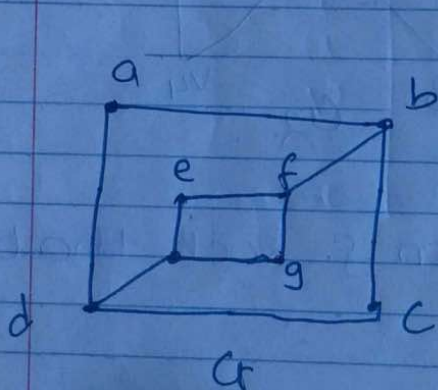


one-to-one & onto

So, the given two graph G_1 & G_2 are isomorphic.



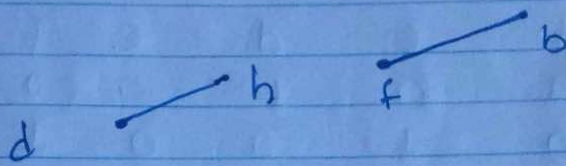
Although there are equal no. of vertices & edges in both the graphs G & H but the degree of vertices of both graphs do not match. So, the given graphs are not isomorphic.



Although there are equal no. of vertices & edges in both graphs. In G $\deg(a)$ in G is 2. So the corresponding vertices of a can be either t, u, x or y in H . The adjacent vertices of a has degree equal to 3. but the adjacent vertices of t, u, x, y is not equal to 3. So, the given graphs G & H are not isomorphic.

Vertices having highest degree in given graph is taken for subgraph.

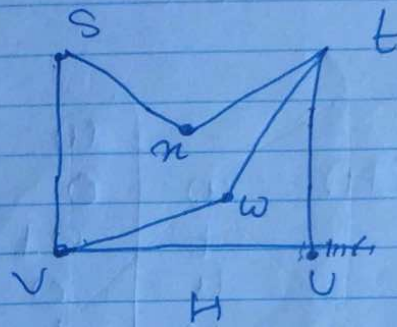
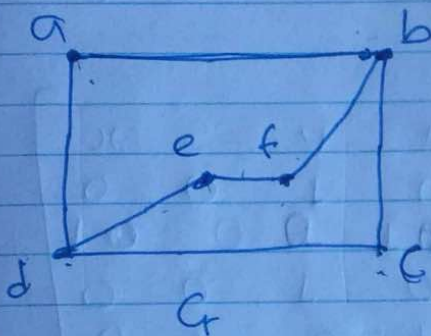
Subgraph of G is



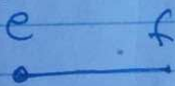
Subgraph of H is



Check whether the given 2 graphs G & H are isomorphic or not.



Subgraph of G



deg 3 at
single
edge is not connected
so deg 2 is ok.

subgraph of H .



Now, let us define a function f such that $f(a) = w$

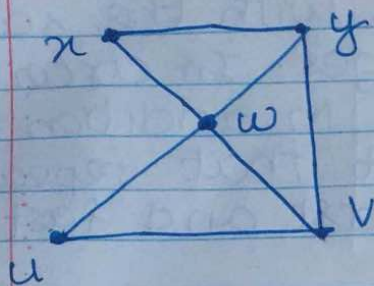
$$\begin{aligned} f(b) &= t \\ f(c) &= u \\ f(d) &= v \\ f(e) &= s \\ f(f) &= x \end{aligned}$$

let the order of vertices in G be a, b, c, d, e, f .

$$A(Cr) = \begin{bmatrix} & a & b & c & d & e & f \\ a & 0 & 1 & 0 & 1 & 0 & 0 \\ b & 1 & 0 & 1 & 0 & 0 & 1 \\ c & 0 & 1 & 0 & 1 & 0 & 0 \\ d & 1 & 0 & 1 & 0 & 1 & 0 \\ e & 0 & 0 & 0 & 1 & 0 & 1 \\ f & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

now, $A(H) = \begin{bmatrix} & w & t & u & v & s & x \\ w & 0 & 1 & 0 & 1 & 0 & 0 \\ t & 1 & 0 & 1 & 0 & 0 & 1 \\ u & 0 & 1 & 0 & 1 & 0 & 0 \\ v & 1 & 0 & 1 & 0 & 1 & 0 \\ s & 0 & 0 & 0 & 1 & 0 & 1 \\ x & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

walks, trails, paths, circuits & cycles



$\text{walk}(W): u, v, w, v, y, w$ ^{open walk}
 $\text{Trail (T)}: x, w, v, u, w, y$ ^{initial & final vertex}
 $\text{Path (P)}: x, w, v, y$
 $\text{Circuit (C)}: v, u, w, x, y, w, v$ ^{diff.}
 $\text{cycle (C')}: v, w, x, y, v$

WALK:

Walk is an alternative sequence of vertices & edges. A walk that begins & ends at same vertex. In walk, there is representation of edges when we traverse from one vertex to another vertex. The length of the walk = no. of edges encountered.
 In above example length of walk = 5. $\odot$

Trail

A trail is a walk that does not pass over the same edge twice. In trail, there is no repetition of edges but vertices can be repeated.

Path

A path is a walk that doesn't include any vertex twice except that it's first vertex might be the same as it's last.

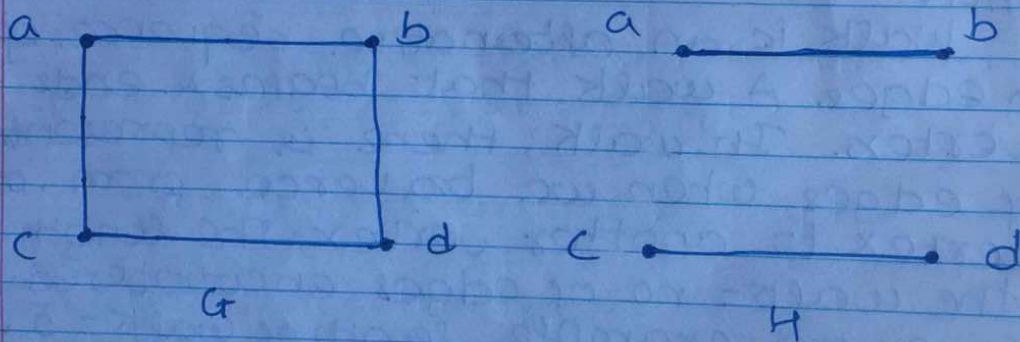
Circuit:

A circuit is a closed trail of length 3 or

more that ends and begins with the same vertex but repeats no edges. In circuit, vertices can be repeated in addition to the first and last. $\rightarrow$ [A circuit that repeats no vertex except for first and last]

Cycle: - - - - -

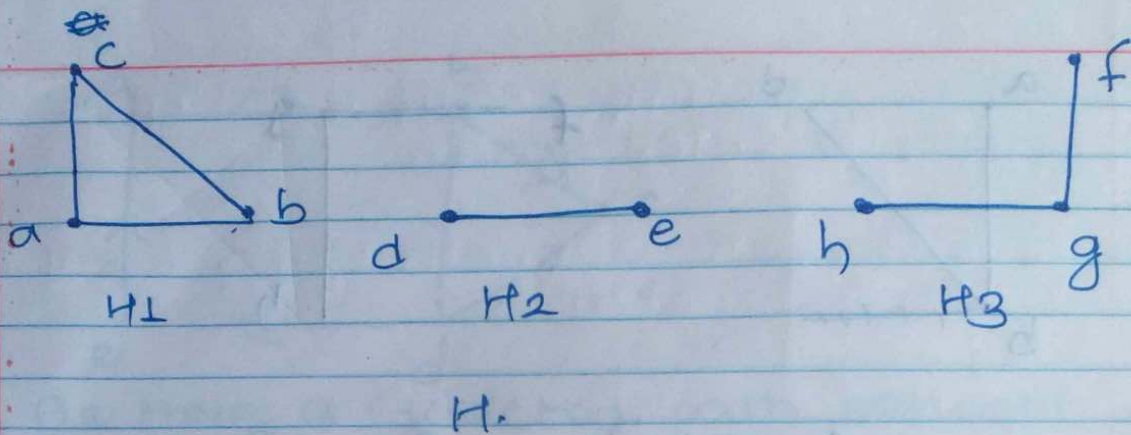
Connectedness in undirected graph:



Here, graph G is connected graph because there exists path between every pair of distinct vertices.

Here Graph H is not connected there doesn't exist a path betⁿ every pair of distinct vertices.

A graph that is not connected is the union of more than one connected graph that donot share the common vertex. This disjoint connected subgraphs are called connected Components of a graph.



$\{a, b, c\}$, $\{d, e\}$, $\{f, g, h\}$

Imp

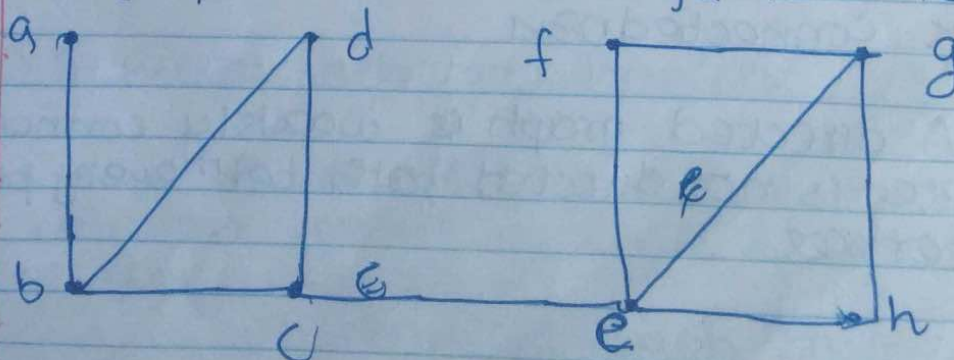
Cut vertices and Cut Edges.

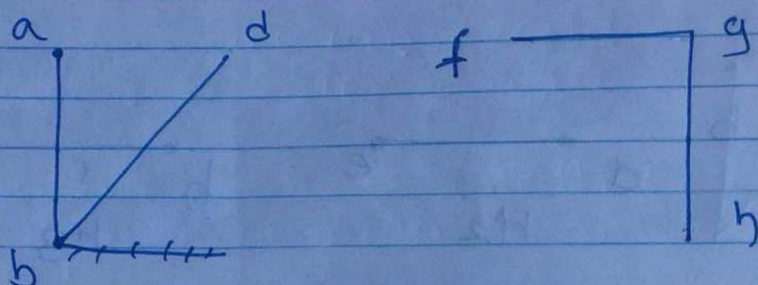
Cut vertex

If the removal of a vertex and all the edges incident with it in a graph produces a subgraph with more connected components than in the original graph, then such vertex is called as the $\#$ cut vertex.

Cut edge

If the removal of an edge creates more connected components than in the original graph then such edge is called cut edge.





cut vertices = ~~$\{c, e\}$~~ c and e.

Cut-edges = $\{c, e\}$

→ Vertex v is cut vertex of G iff $\deg(v)$ is greater than and equal to 2. i.e. $\deg(v) \geq 2$.

Connectedness in directed graph.

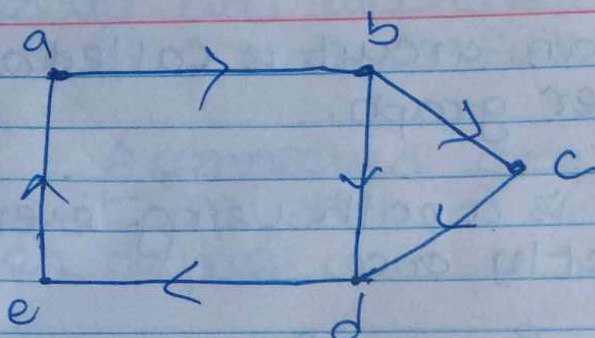
- Strong Connectedness
- Weak Connectedness

Strong Connectedness

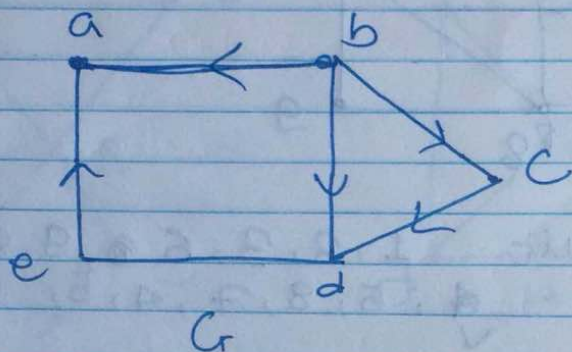
A directed graph is strongly connected if there is a directed path betⁿ every pair of vertices. A ~~graph~~

Weak Connectedness

A directed graph is weakly connected if there is no directed path betⁿ every pair of vertices.



As there is directed path between every pair of distinct vertices so the given ^{directed} graph G is strongly connected.



As there is no directed path between every pair of distinct vertices so the given directed graph G is weakly connected.

If we ignore the directed, if the graph is connected then, it is weakly connected.

Imp

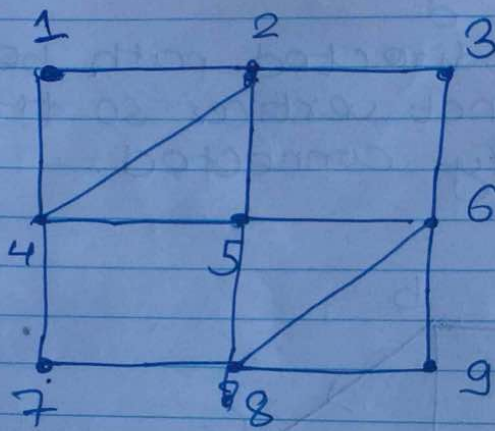
If Euler path and circuits:

A circuit C in a graph G is called Eulerian circuit or Euler's circuit if C contains every edge of G .

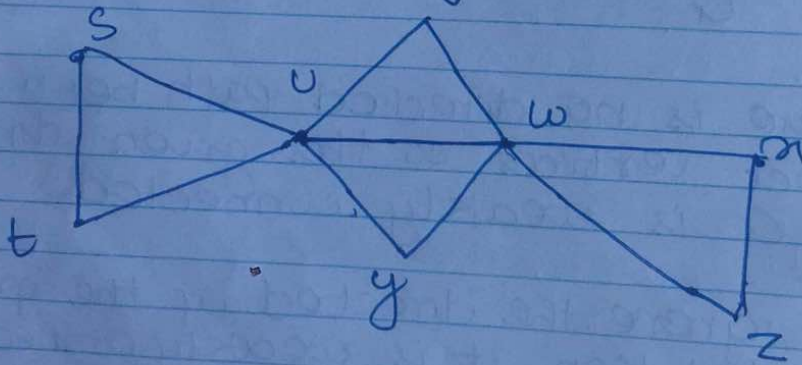
A connected graph that contains an

~~Euler~~ Eulerian circuit is called as Eulerian graph or Euler graph.

→ An Euler path is a path using every edge of Graph exactly once.



Euler circuit: 1, 2, 3, 6, 9, 8, ~~5, 8, 7~~
~~4~~, 5, 8, 7, 4, 5, 2, 4, 1.

ऊ.

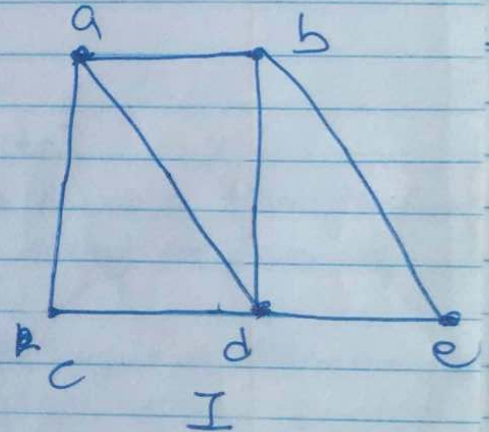
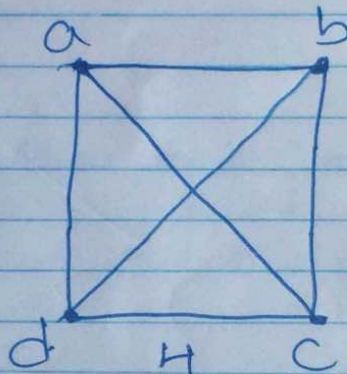
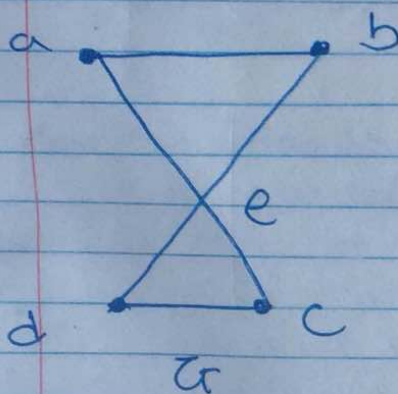
Euler path: $u, s, t, u, v, w, x, z, w, y, u$

Euler circuit: $u, t, s, u, y, w, z, x, w, v, u, w$
path

Necessary and sufficient conditions for Euler circuit & Euler path.

A connected graph G has a Euler circuit iff each of its vertices has an even degree.

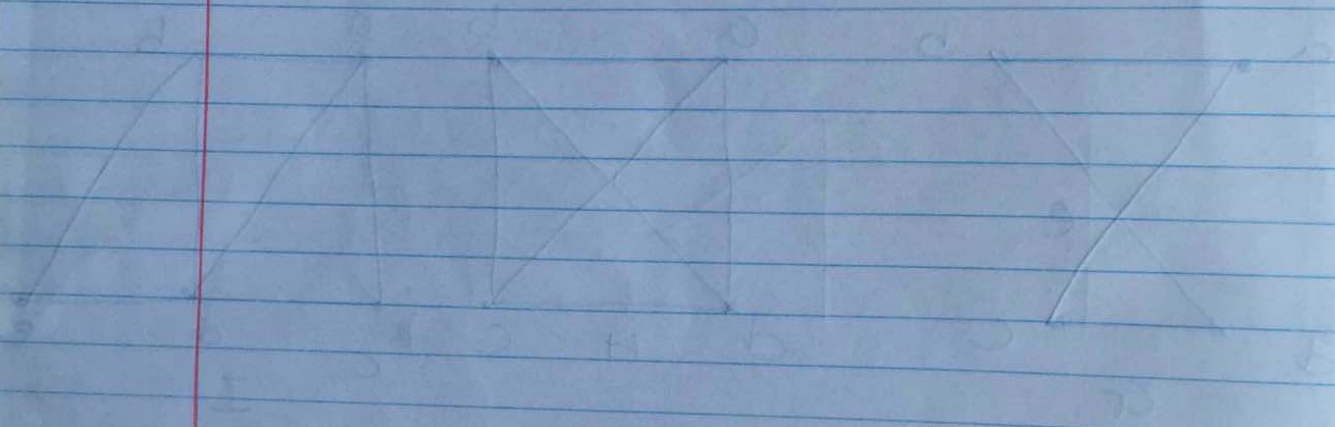
A connected graph G has a Euler trail or path iff exactly two vertices of G has odd degree.



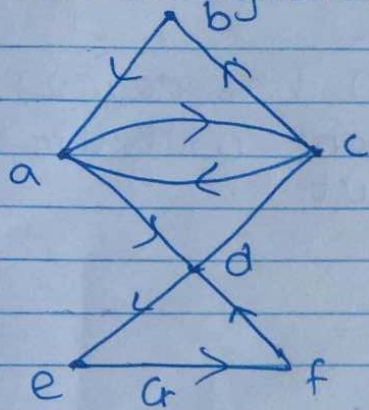
A connected graph is called a tree if it has no cycles.
 A tree with n vertices has $n-1$ edges.

A connected graph with n vertices and $n-1$ edges is a tree.

A connected graph with n vertices and $n-1$ edges is a tree.



A non-trivial connected directed graph (di-graph) is eulerian iff each vertex has the same no. of indegrees and out degrees.

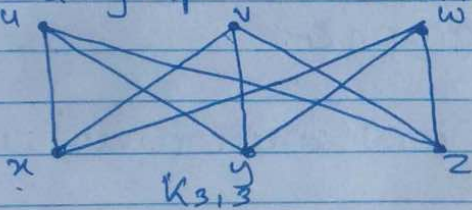


Euler Circuit: a, d, e, f, d, c, b, a

Imp Hamilton (Hamiltonian) Paths and Circuits:

A connected graph G is said to have a Hamiltonian path if there exist a path that covers every vertices of a graph G exactly once.

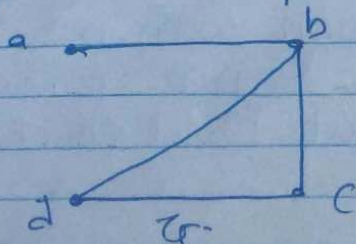
A connected graph G is said to have a Hamiltonian circuit if there exist a circuit that covers every vertices of a graph G exactly once.



Hamilton Circuit: u, x, v, y, w, z, u

Hamilton Path: u, y, v, z, w, x

If there is presence of Hamiltonian Circuit in any Graph G then automatically there is a presence of Hamilton path but the converse is not true.

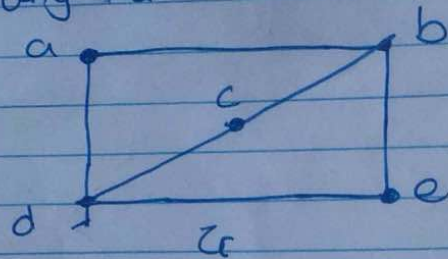


Here, there is Hamiltonian path but not Hamiltonian circuit

Conditions for Hamiltonian graphs (Properties that can be used to show a graph has no Hamiltonian circuit)

Properties:

- i) A graph with a vertex of $\deg(v) < 2$ cannot have a Hamiltonian circuit.
- ii) If a vertex in the graph has $\deg(v) = 2$ then both edges that are incident with this vertex must be in any Hamiltonian circuit.



In the above graph G , vertex c has $\deg(c) = 2$, the incident edges for this vertex are $\{cb\}$ and $\{cd\}$. In order to be present in a Hamiltonian circuit, in this graph G , these edges must be part of any Hamiltonian circuit which is not possible so there is no presence of Hamiltonian circuit in graph G .

Sufficient conditions for Hamiltonian circuit.

- Dirac's Theorem
- Ore's Theorem

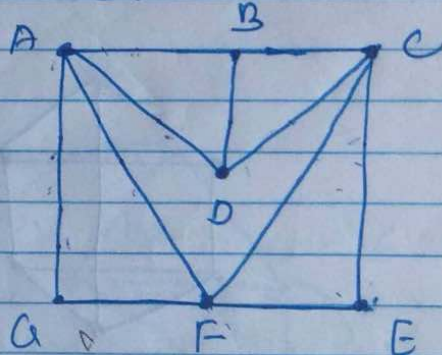
→ Dirac's Theorem:

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in graph G is at least $n/2$ then G has a Hamiltonian circuit.

→ Ore's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u, v in G then G has a Hamiltonian circuit.

Find the Euler and Hamilton circuit of the given graph.



Here, in the graph G , there is no Euler circuit because the vertex degree of every vertex is not even.

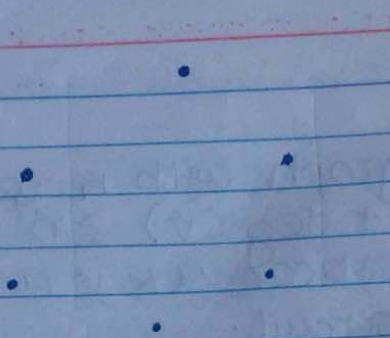
Also, though the necessary condition is not fulfilled in case of Hamiltonian circuit, ~~there~~ in given graph G , there is presence of Hamiltonian circuit since it can be started with one vertex and ended with the same vertex, with repeating edges.

A, D, C, E, F, G, A

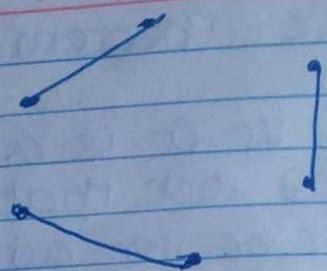
Imp

Regular Graph

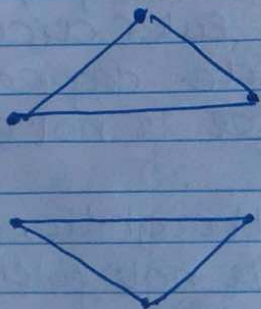
Any graph G is said to be regular if each vertices of graph G has same degree. A regular graph with vertices ' K ' is said to be K -regular graph or regular graph of degree K .



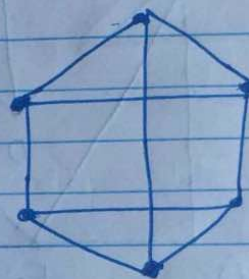
0-Regular graph



1-Regular graph



2-Regular graph

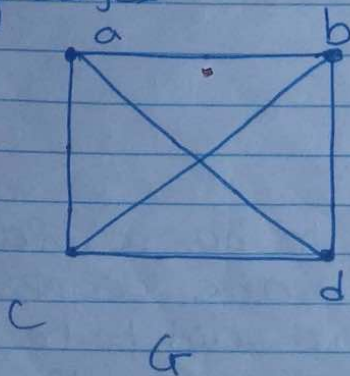


3-Regular graph

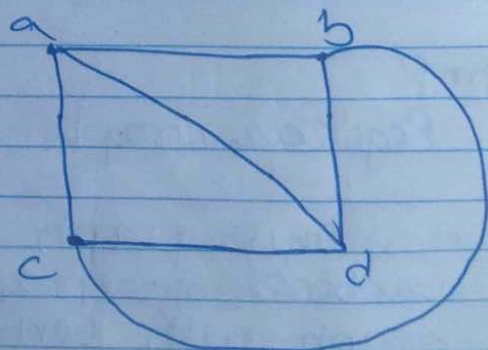
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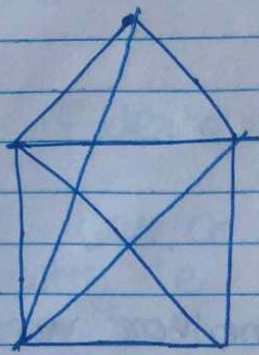
Planar Graph

Any graph G is said to be planar if it can be redrawn in a plane without crossing ^{any} edges.

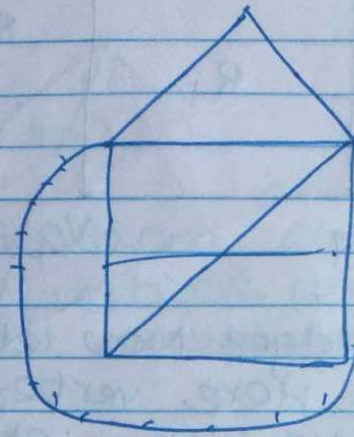


$\Rightarrow$

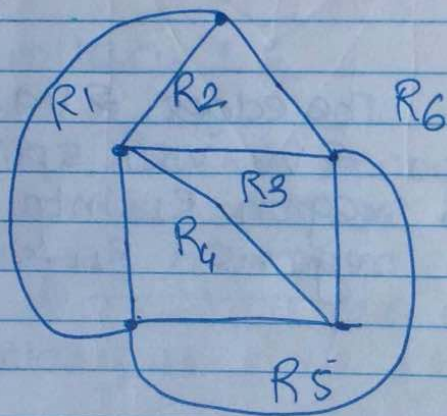




$\Rightarrow$

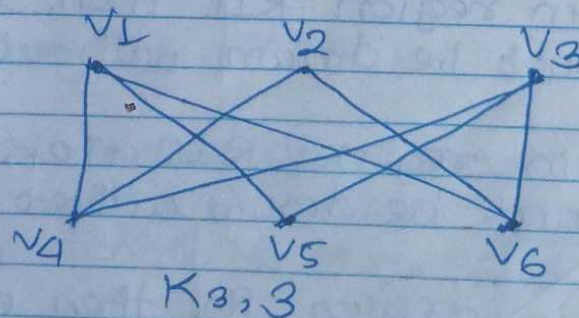


$\Downarrow$

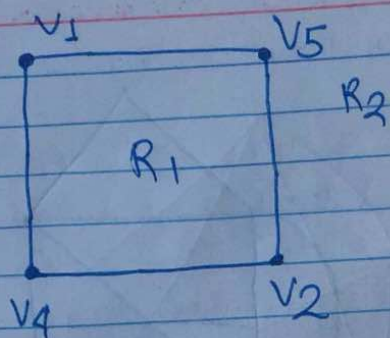


Imp

Is $K_{3,3}$ a planar graph?

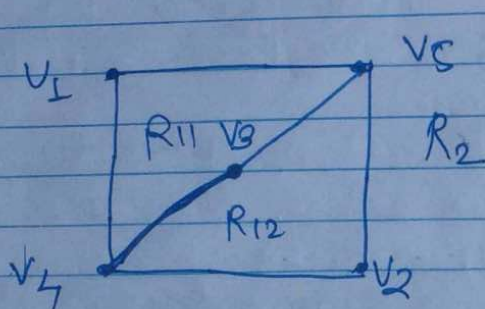


First of all consider four vertices V_1, V_2, V_4 & V_5 . The edges $\{V_1, V_4\}$, $\{V_4, V_2\}$, $\{V_2, V_5\}$ & $\{V_5, V_1\}$ form a closed curve that splits the



the plane into two region R_1 & R_2 .

The edges Now let us consider another vertex V_3 . Here, vertex V_3 is connected to vertices V_4 , V_5 & V_6 . let us consider vertex V_3 lies in region R_1 .



The edges $\{V_4, V_3\}$, $\{V_3, V_5\}$ splits the region R_1 into two regions: R_{11} & R_{12} .

Now let us consider the remaining vertex V_6

- If V_6 lies in region R_{11} then, the edge $\{V_6, V_2\}$ cannot be drawn without crossing edges.
- If V_6 lies in region R_{12} then, the edge $\{V_6, V_2\}$ cannot be drawn without crossing.
- If V_6 lies in region R_2 then edge $\{V_6, V_2\}$ cannot be drawn without crossing.

Hence $K_{3,3}$ cannot be drawn in a plane without crossing any edges so the given

graph $K_{3,3}$ is non-planar.

Euler's Formula (Theorem)

Let G be a connected planar simple graph with ' e ' edges and ' v ' vertices. Let R be the no. of regions in a planar representation of G . Then,

$$r = e - v + 2$$

Corollary 1:

If G is a connected planar graph with ' e ' edges and ' v ' no. of vertices where $v \geq 3$ then $\boxed{e \leq 3v - 6}$

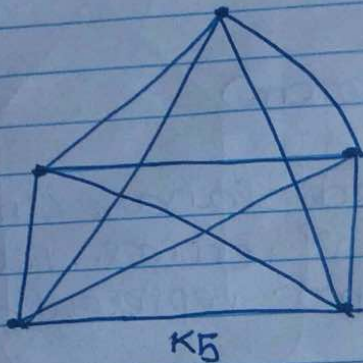
Corollary 2:

If G is a connected planar simple graph then G has a vertex of degree ≤ 5 not exceeding 5.

Corollary 3:

If a connected planar simple graph has ' e ' edges and ' v ' vertices with $v \geq 3$ and no circuits of length 3 then $\boxed{e \leq 2v - 4}$.

Is K_5 a planar graph?



Here, no. of vertices (V) = 5
no. of edges (E) = 10.

From corollary I, if K_5 is planar

$$E \leq 3V - 6$$

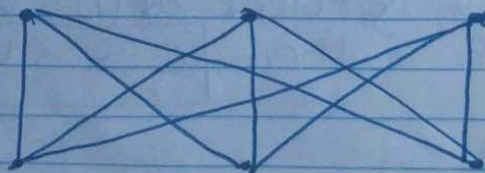
$$\Rightarrow 10 \leq 3 \times 5 - 6$$

$$\Rightarrow 10 \leq 15 - 6$$

$$\Rightarrow 10 \leq 9. \text{ which is not true}$$

So, the figure given figure K_5 is not a planar graph.

Is $K_{3,3}$ a complete bipartite graph?
planar



Here, no. of vertices (V) = 6
no. of edges (E) = 9.

If $K_{3,3}$ is planar then, according to Corollary I,

$$e \leq 3v - 6$$

$$9 \leq 3 \times 6 - 6$$

$$9 \leq 12 \text{ which is true.}$$

Again,

Now, According to Corollary II, if $K_{3,3}$ is planar then the degree of vertices of $K_{3,3}$ should not exceed 5 which is true. i.e. $3 \leq 5$.

Again, if $K_{3,3}$ is planar, then $K_{3,3}$ should not have circuit of length 3 which is true in this case.

Again, if $K_{3,3}$ is planar then the condition $e \leq 2v - 4$ should be satisfied

$$\text{Here } 9 \leq 2 \times 6 - 4$$

$$\Rightarrow 9 \leq 8 \text{ which is false.}$$

So, the given graph $K_{3,3}$ is non-planar.

Imp Dijkstra's Algorithm:

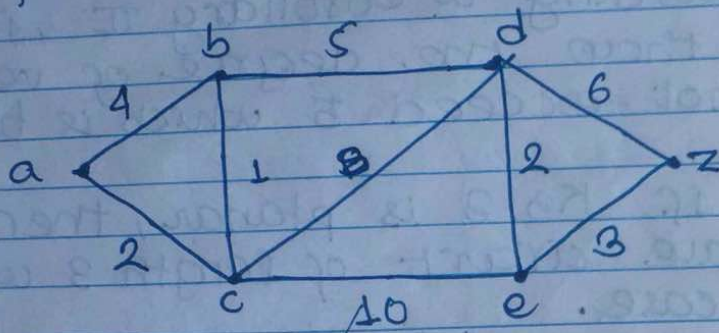
One of the algorithms to find the length of the shortest path from one vertex to another vertex in a weighted graph. Here, initial vertex is known as source vertex and terminal vertex is known as destination vertex.

Weighted graph.: A graph that is assigned a number to each edge is called weighted graph.

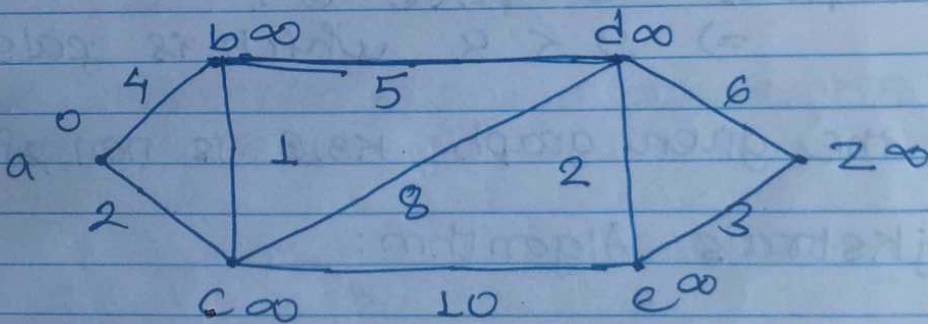
→ In a weighted graph, the length of the path is the sum of the weight of the edges.

Imp

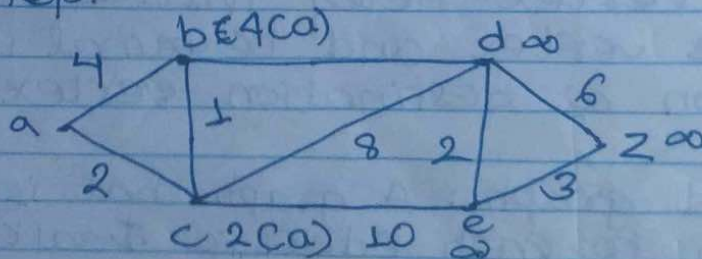
Use Dijkstra's Algorithm to find the length of the shortest path from the vertices a to z in the following weighted graph.



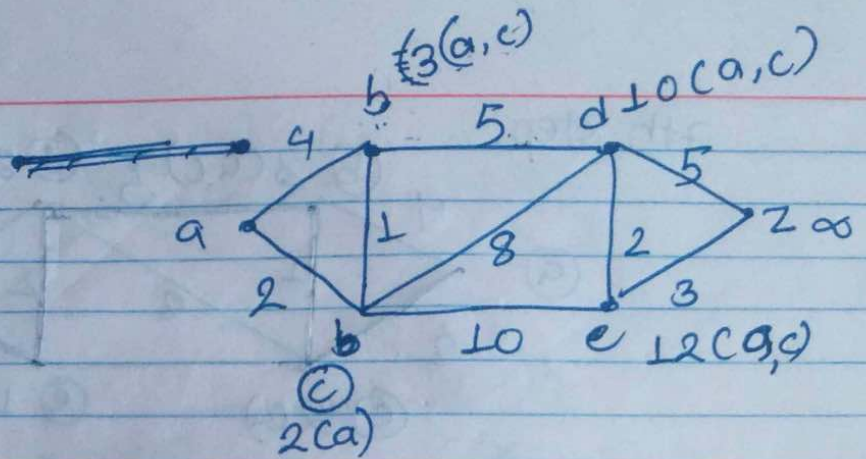
1st step: Initialize ^{source} 1st vertex as zero and remaining vertex to infinite symbol.



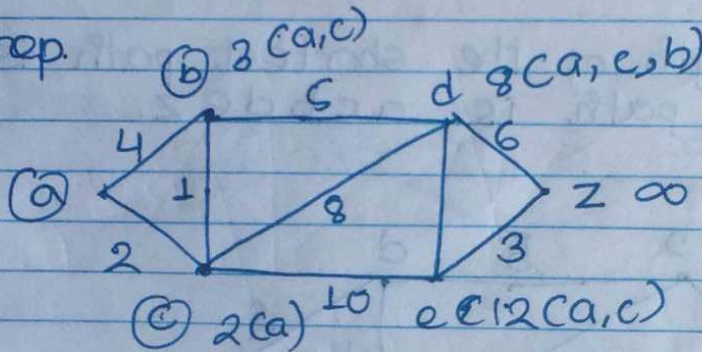
2nd step:



3rd step:

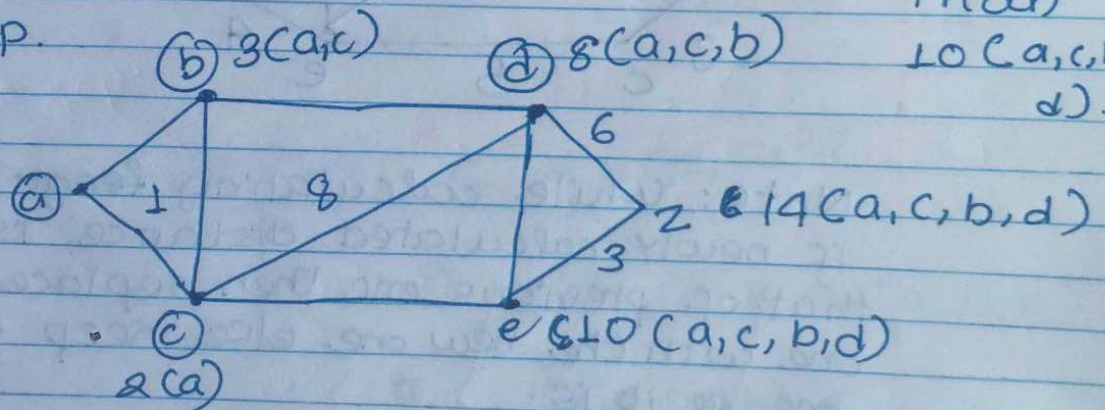


4th step.

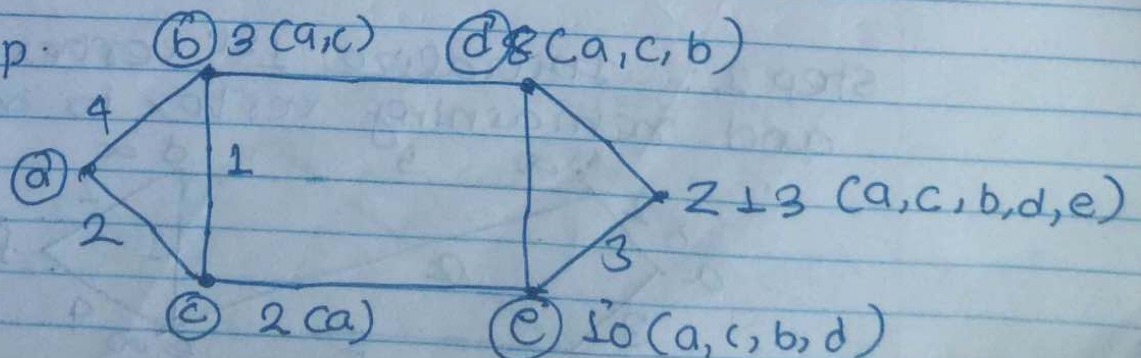


$12(a,c)$ is more than $10(a,c,b,d)$.

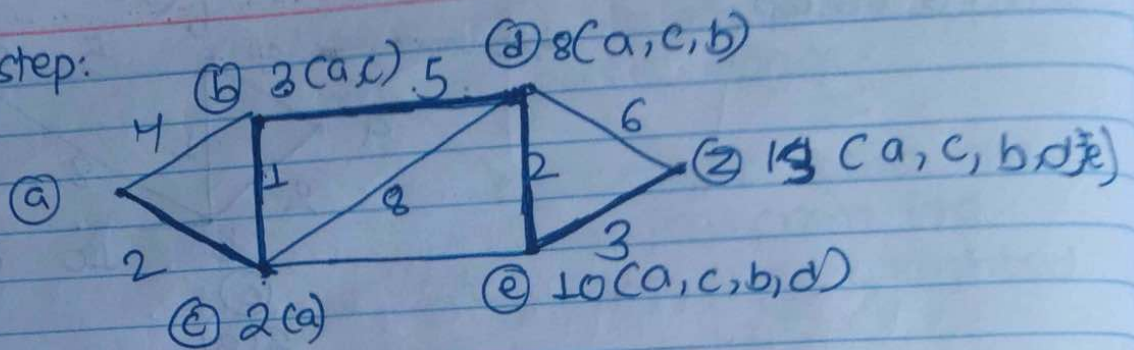
5th step.



6th step.

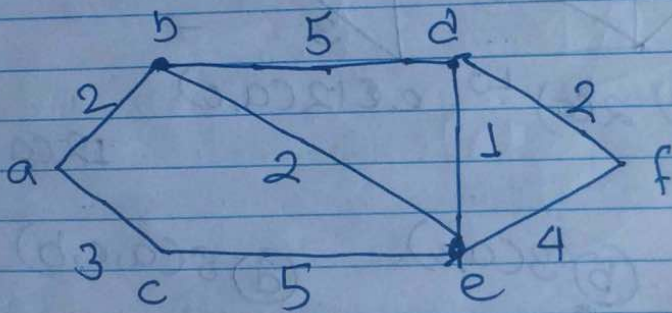


7th step:



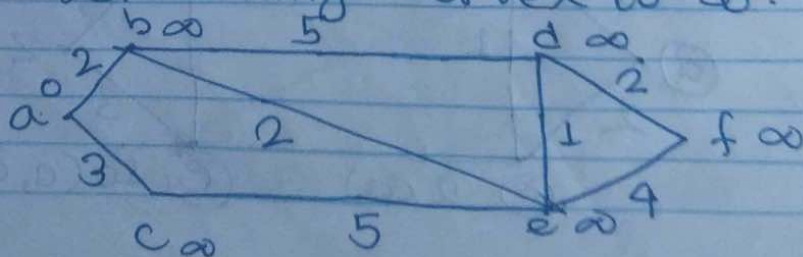
Here, length of the shortest path is 13 and the path is a c b d e z.

2)

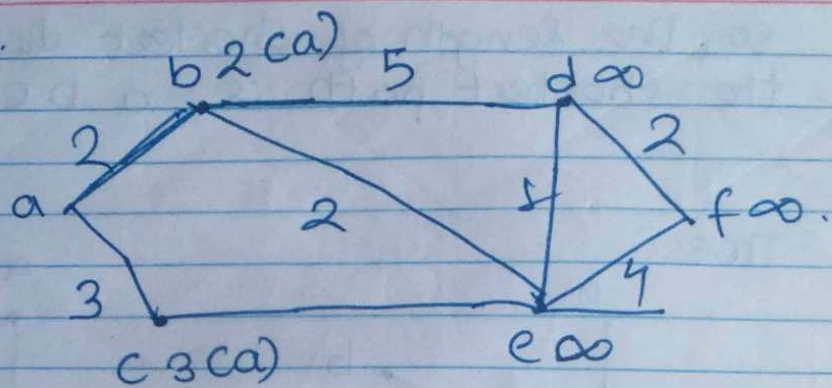


Note: While calculating length distance, if newly calculated distance is less than that of previous one then replace the previous one with the new one else keep previous one as it is.

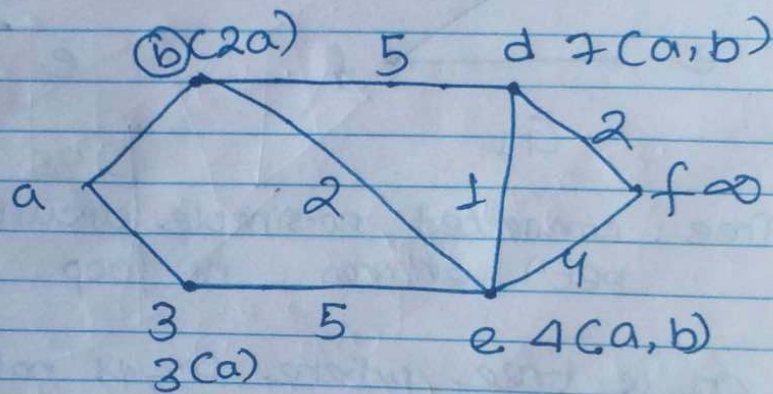
Step 1: Initialize 1st vertex as zero and remaining vertex to ∞ .



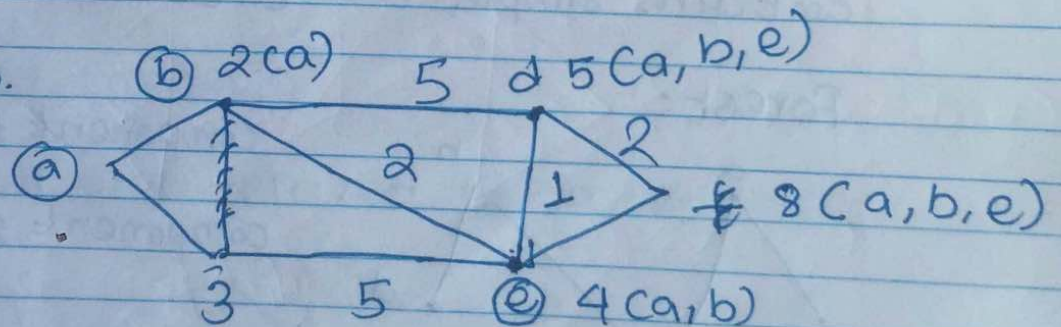
2nd step.



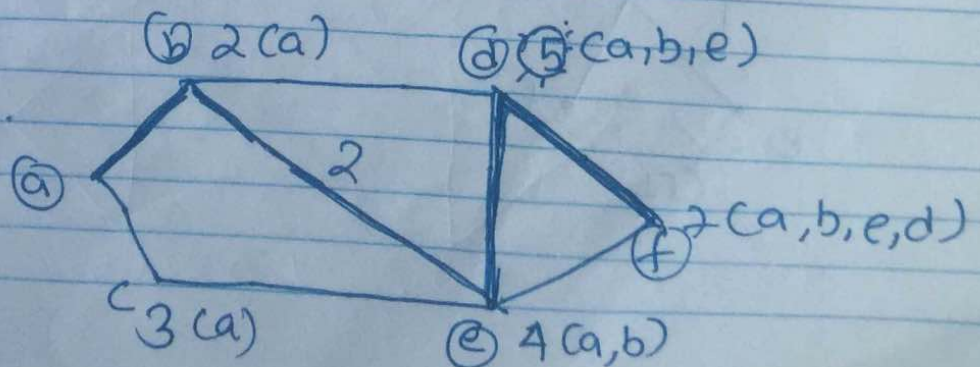
3rd step:



4th step.

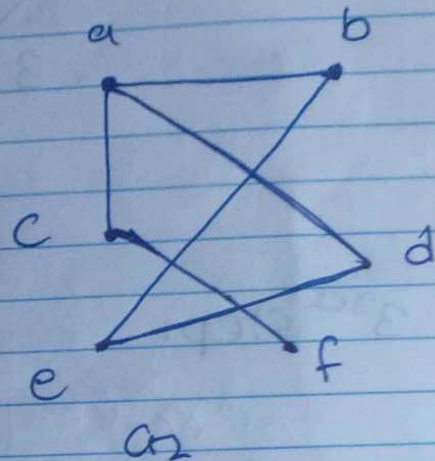
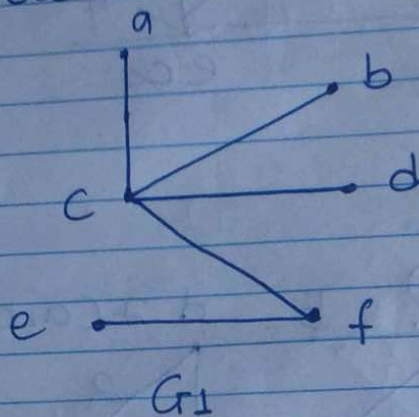


5th step.



So, the length of shortest distance is 7 and the shortest path is a b e d f

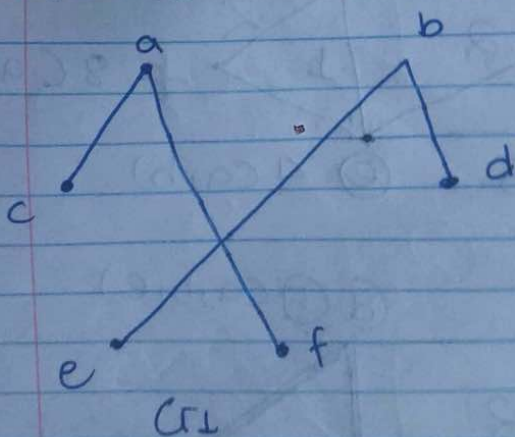
Trees:



Tree: connected, no-simple circuit, no multiple edges betⁿ vertices, no loop

G_1 is tree where G_2 is not a tree as G_2 contains simple ~~tree~~ circuit.

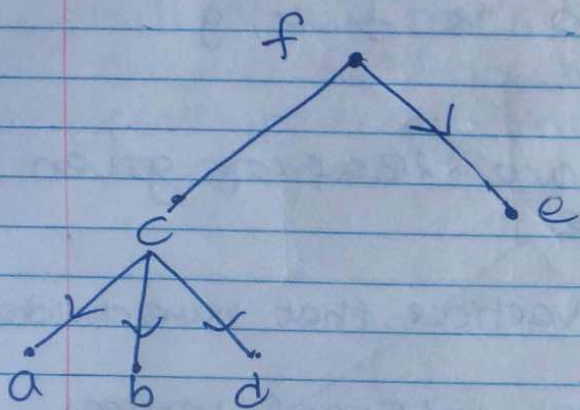
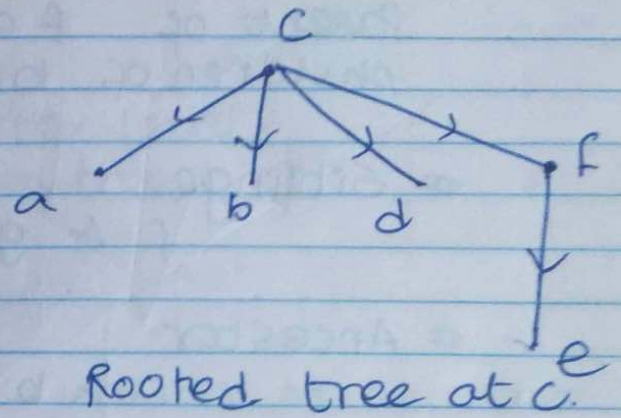
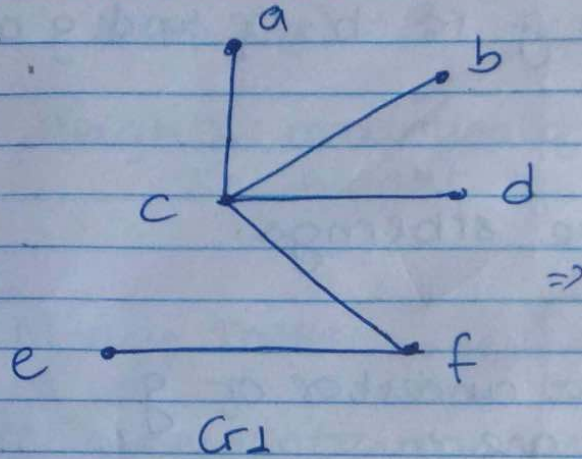
Forest:



Component 1: a, c, e, f

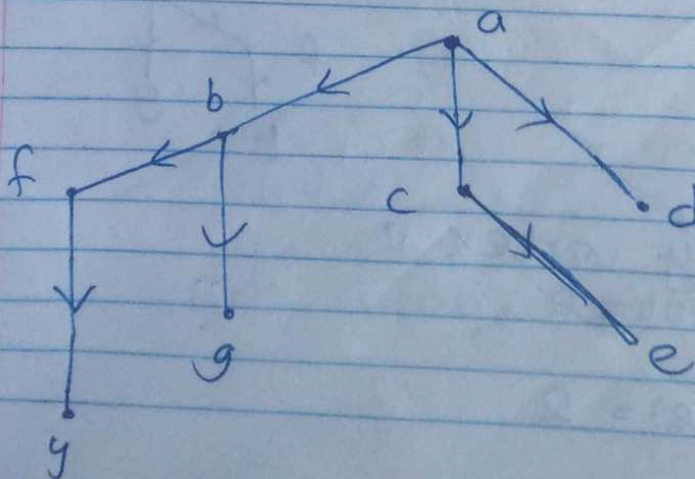
Component 2: b, d, f

Rooted (directed) tree:



Rooted tree at f .

Terminologies related to rooted tree:



Parent of f and g is b . f and g are children of b .

Siblings
 f & g are siblings.

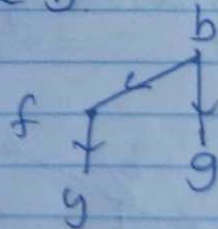
Ancestor
 a & b are ancestor of g
 f , b & a are " " " g

Descendants:
Descendants of b is f, g, g .

Leaf:
 y, g, e & d are leaf of given rooted tree

Internal vertices: Vertices that have children
 a, b, c, f are internal vertices.

Sub-tree
sub-tree rooted at vertex b .



Level of any vertex v
level of root = 0
level(v)
level(e) = 2

level (y) = 3

Height : maximum of vertex level
so height of given tree = 3

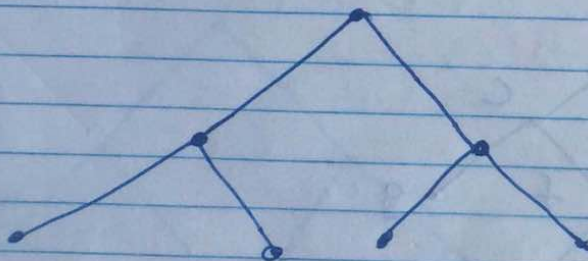
Imp

M-ary Trees :

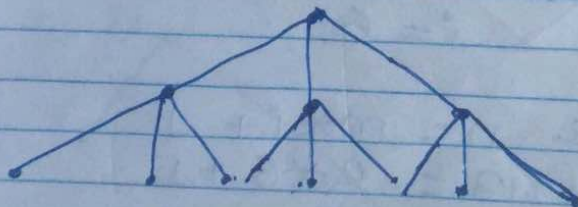
A rooted tree is called M-ary tree if every internal vertex has no more than M children.

Full M-ary trees:

Any M-ary tree is said to be full if every internal vertex has exactly m children.

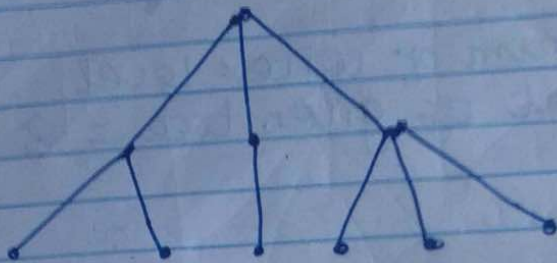


Full binary / 2-ary tree



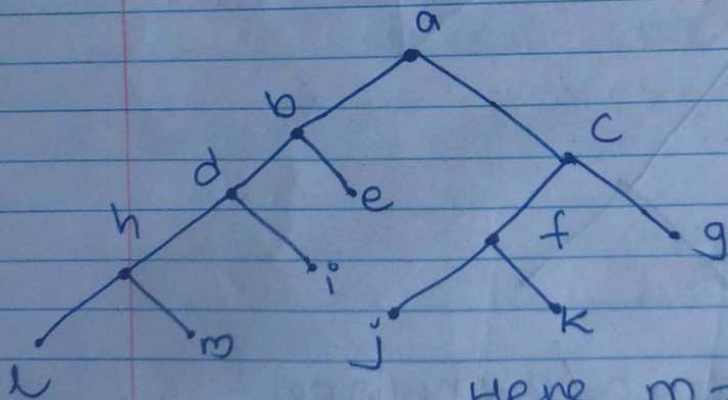
Full 3-ary tree

3-ary tree



Properties of Trees:

- A tree with n - vertices has $n-1$ edges.
- A full m -ary tree with ' i ' internal vertices contain $n = m*i + 1$ vertices.



Here $m = 6$
 $n = 13$
 $i = 6$

We have $n = m*i + 1$
 $13 = 2*6 + 1$
 $\Rightarrow 13 = 13$

#

• A full m-ary tree with

i) n vertices

$i = (n-1) / m$ internal vertices &

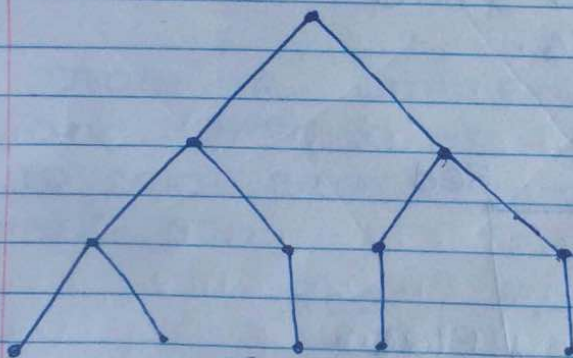
$l = [(m-1)n + 1] / m$ leaves

ii) l -leaves has $n = (ml - 1) / (m-1)$ vertices

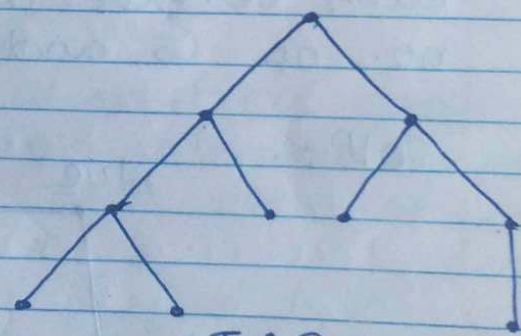
& $i = (l-1) / (m-1)$ internal vertices

Balanced tree:

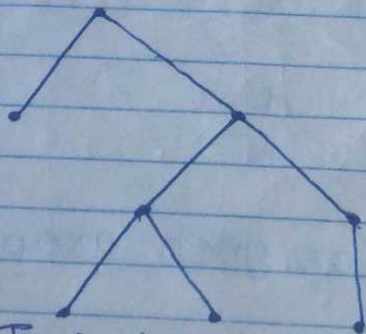
A rooted m-ary tree of height h is balanced if all leaves are at level h or $h-1$.



T_1 : Balanced

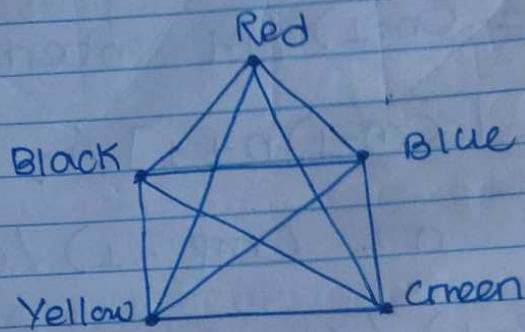


T_2 : Balanced



T_3 : Not Balanced

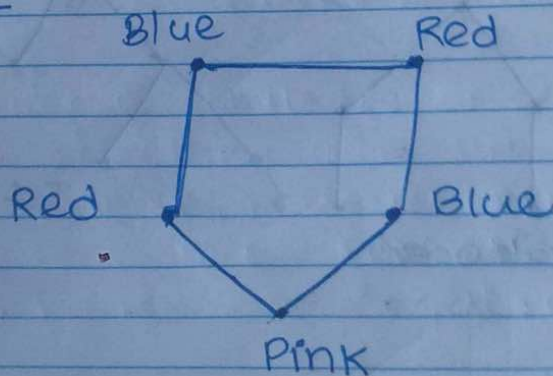
Imp Draw the complete graph K_5 and use graph coloring techniques to color each vertices and find the chromatic number of the given graph?



So the chromatic number of given graph is 5.

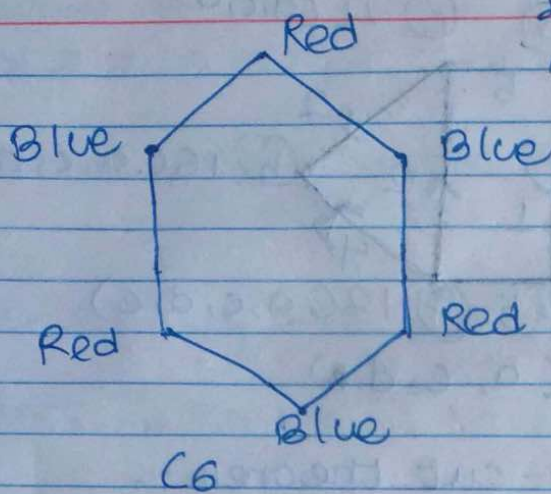
Draw the cycle graph C_5 and C_6 and use the graph coloring techniques to color each vertices and find the chromatic no. of C_5 and C_6 .

Solⁿ:-



C_5

The chromatic no. of given graph C_5 is 3.



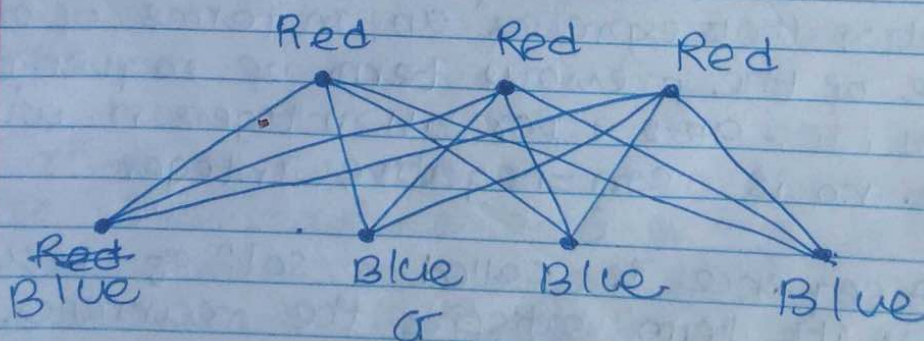
In cycle graph, if the no. of vertices is even chromatic no. is also even.

So, the chromatic no. of C_6 is 2

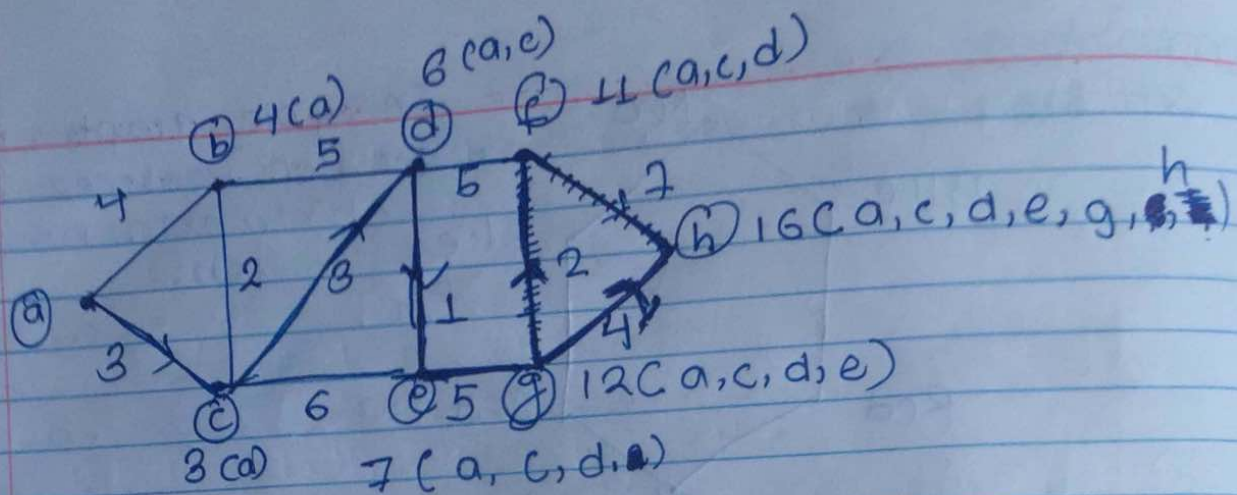
$$\chi(C_5) = 3$$

$$\chi(C_6) = 2.$$

Draw the complete bi-partite graph $K_{3,4}$ and use the graph coloring technique to color each vertices and find the chromatic no.

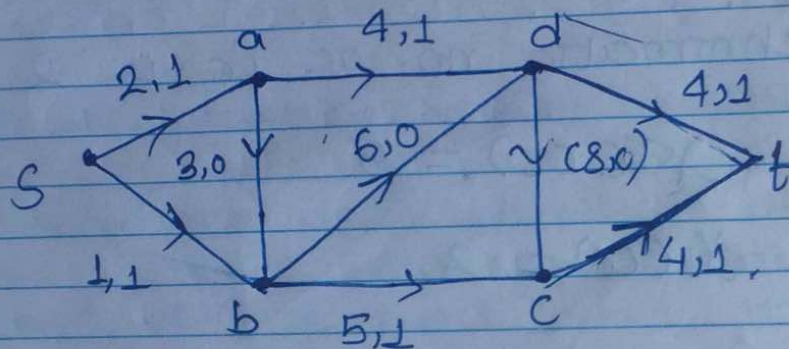


$$\chi(G) = 2$$



Imp

Max. flow Min-cut theorem:-



chapter 3 (Recurrence Relation.) 8 marks

Recurrence relation for a sequence a_n is an equation that expresses a_n in terms of one or more of the previous terms of sequence namely $a_0, a_1, \dots, a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is non-negative integer.

A sequence is called a solⁿ of recurrence relⁿ if its term satisfies the recurrence relation.

$$a_n = 2 * a_{n-1}$$

solution of the linear homogeneous recurrence relation.

Standard form (Linear homogeneous recurrence relation)

The eqn of the form: where $a_n = c_1 a_{n-1}$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \quad \text{--- (1)}$$

Solving linear homogeneous recurrence relation

Basically when solving ^{linear homogeneous} recurrence relⁿ, we substitute $a_n^* = r^n$ where r is a constant

So, eqn (1) becomes

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + c_3 r^{n-3} + \dots + c_k r^{n-k}$$

Dividing both sides by r^{n-k} , k is the highest degree

$$\boxed{r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k} \quad \text{characteristic eqn.}$$

Theorem:

Let c_1, c_2 be the real no. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 & r_2 . The sequence $\{a_n\}$ is the solⁿ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, iff $a_n = a_1 r_1^n + a_2 r_2^n$

Eg:

$$a_n = a_{n-1} + 2a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 7$$

$$\text{put } a_n = r^n$$

$$r^n = r^{n-1} + 2r^{n-2}$$

Dividing both sides by r^{n-2}

$$r^2 = r + 2$$

$$r^2 - r - 2 = 0$$

$$r^2 - 2r + r - 2 = 0$$

$$r(r-2) + 1(r-2) = 0$$

$$(r-2)(r+1) = 0$$

$$\therefore r = 2, -1$$

$$r_2 = -1 \text{ \& } r_1 = 2$$

$$\text{Now, } a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$\text{When } n=0, \quad a_0 = \alpha_1 r_1^0 + \alpha_2 r_2^0$$

$$n=0$$

$$2 = \alpha_1 + \alpha_2 \quad \text{--- (i)}$$

$$\text{When } n=1,$$

$$a_1 = \alpha_1 r_1 + \alpha_2 r_2$$

$$7 = \alpha_1 \times 2 + \alpha_2 \times (-1)$$

$$7 = 2\alpha_1 - \alpha_2 \quad \text{--- (ii)}$$

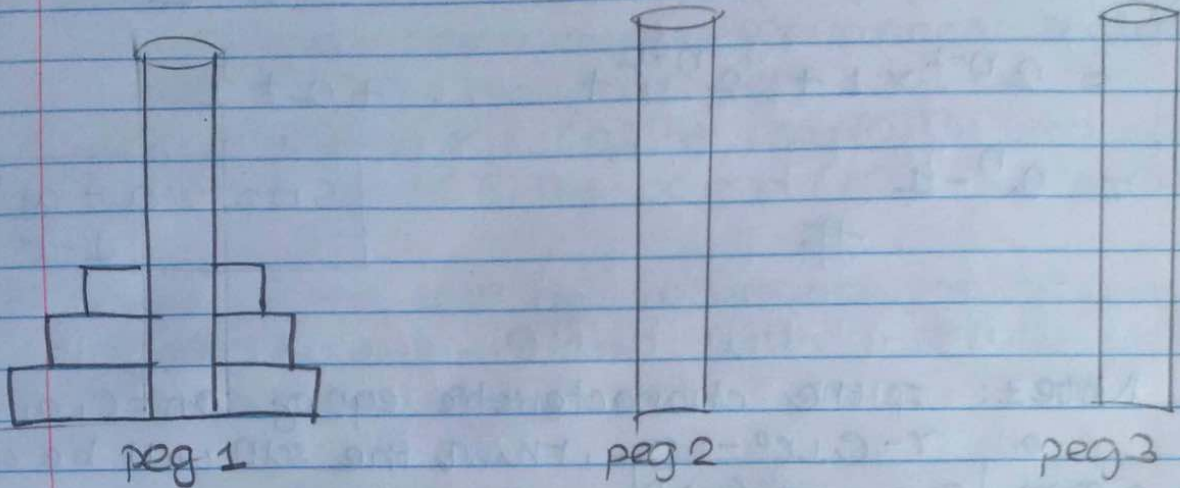
$$\text{Solving (i) \& (ii), } \alpha_1 = 3$$

$$\alpha_2 = -1$$

##

$$\text{So, the required soln, is } a_n = 3r_1^n - r_2^n$$

Recurrence Relation for TOH (Tower of Hanoi) problems:



let H_n denotes the no. of moves needed to ~~transfer~~ solve the TOH problem with 'n' ~~bits~~ disks. let us begin with n disks in peg 1.

First, we move $(n-1)$ disk to peg 3 which takes H_{n-1} moves. Now we move the largest disk from peg 1 to peg 2 which takes 1 move. Finally, we move $(n-1)$ disk from peg 3 to peg 2 which takes H_{n-1} moves.

The initial condition is $H_1 = 1$ because 1 disk can be transferred from peg 1 to peg 2 in 1 move.

$$\begin{aligned}\text{Here Total moves } (H_n) &= H_{n-1} + 1 + H_{n-1} \\ &= 2H_{n-1} + 1 \\ &= 2(2H_{n-2} + 1) + 1 \\ &= 2^2 H_{n-2} + 2 + 1 \\ &= 2^2 (2H_{n-3} + 1) + 2 + 1 \\ &= 2^3 H_{n-3} + 2^2 + 2 + 1 \\ &\vdots\end{aligned}$$

$$2^{n-1} H_{n-(n-1)}$$

$$= 2^{n-1} H_1 + 2^{n-2} + \dots + 2 + 1$$

$$= 2^{n-1} \times 1 + 2^{n-2} + \dots + 2 + 1$$

$$= 2^n - 1$$

#

$$S_n = \frac{a - ar^n}{1-r}$$

Note 1: If the characteristic eqn of $a_n = C_1 a_{n-1}$ is $r - C_1 r^0 = 0$, then the soln will be of the form $a_n = \alpha_1 (r)^n$

Note 2: If the characteristic eqn of $a_n = C_1 a_{n-1} + C_2 a_{n-2}$ is $r^2 - C_1 r - C_2 = 0$ has two distinct roots say r_1 & r_2 then the soln will be of the form, $a_n = \alpha_1 (r_1)^n + \alpha_2 (r_2)^n$ where α_1 & α_2 are constant. If the characteristic eqn has 2 roots which are same, ^{say $r_1 = r_2 = r$} then the soln will be of the form.

$$a_n = \alpha_1 (r)^n + \alpha_2 n (r)^n$$

Note 3: If the characteristic eqn of $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_K a_{n-K}$ is $r^K - C_1 r^{K-1} - C_2 r^{K-2} - \dots - C_K r^0$ has K distinct roots then the soln will be of the form $a_n = \alpha_1 (r_1)^n + \alpha_2 (r_2)^n + \alpha_3 (r_3)^n + \dots + \alpha_K (r_K)^n$

Note 4: Suppose that the roots of the characteristic eqn of linear homogeneous recurrence relation are 2, 2, 2, 5, 5, 9 then the soln will be of the form

$$a_n = \alpha_1 (2)^n + \alpha_2 n (2)^n + \alpha_3 n^2 (2)^n + \alpha_4 (5)^n + \alpha_5 n (5)^n + \alpha_6 (9)^n$$

What is the soln of recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions with $a_0 = 1$ and $a_1 = 6$.

Soln:- Substitute $a^n = r^n$

$$r^n = 6r^{n-1} - 9r^{n-2} \quad 6r^{n-1} - 9r^{n-2}$$

Dividing both sides by r^{n-2} .

$$r^2 = 6r - 9$$

$$\Rightarrow r^2 - 6r + 9 = 0$$

$$\Rightarrow r^2 - 3r - 3r + 9 = 0$$

$$\Rightarrow r(r-3) - 3(r-3) = 0$$

$$\Rightarrow (r-3)(r-3) = 0$$

$$\therefore r = 3, 3 \Rightarrow r_1 = r_2 = r = 3.$$

$$a_n = \alpha_1 r^n + \alpha_2 n r^n$$

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

Put $n = 0$,

$$a_0 = \alpha_1$$

$$\alpha_1 = 1$$

Put $n = 1$,

$$a_1 = \alpha_1 3 + \alpha_2 3$$

$$6 = 1 \times 3 + 3\alpha_2 \Rightarrow 3\alpha_2 = 3$$

$$\alpha_2 = 1$$

$$\text{So } \alpha_1 = 1 \\ \alpha_2 = 1.$$

So, the required soln is $a_n = 3^n + n3^n$
 $= 2^n C_{n+1} \#$

- Q) Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial conditions $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$.

Sol: Given recurrence reln is

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

Substitute $a_n = r^n$

$$r^n = 6r^{n-1} - 11r^{n-2} + 6r^{n-3}$$

Now, Dividing both sides by r^{n-3}

$$r^3 = 6r^2 - 11r + 6$$

$$\Rightarrow r^3 - 6r^2 + 11r - 6 = 0$$

$$\therefore r = 1, 2, 3$$

$$\text{So } a_n = C_1 \alpha_1 (1)^n + \alpha_2 (2)^n + \alpha_3 (3)^n$$

$$\text{Put } n=0, a_0 = \alpha_1 + \alpha_2 + \alpha_3 \quad \text{--- (i)}$$

$$2 = \alpha_1 + \alpha_2 + \alpha_3 \quad \text{--- (i)}$$

$$\text{Put } n=1, a_1 = 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3 \quad \text{--- (ii)}$$

Put $n=2$, $15 = \alpha_1 + 4\alpha_2 + 9\alpha_3$ — (iii)

solving (i), (ii) & (iii),

$$\alpha_1 = 1$$

$$\alpha_2 = -1$$

$$\alpha_3 = 2$$

So the required soln is.

$$\begin{aligned} a_n &= \alpha_1 (1)^n + \alpha_2 (2)^n + \alpha_3 (3)^n \\ &= 1^n - 2^n + 2 \cdot 3^n \end{aligned}$$

Q) Find the recurrence relation for fibonacci series.

0, 1, 1, 2, 3,

$a_n = a_{n-1} + a_{n-2}$ with initial conditions $a_0 = 0$
 $a_1 = 1$

Substitute $a_n = x^n$

$$x^n = x^{n-1} + x^{n-2}$$

Dividing both sides by x^{n-2}

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

$$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times -1}}{2 \times 1}$$

$$\therefore r = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow r_1 = \frac{1 + \sqrt{5}}{2} \text{ \& } r_2 = \frac{1 - \sqrt{5}}{2}$$

So, The required soln is

$$a_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n \alpha_1 + \alpha_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

using initial condition, $a_0 = 0$
 $a_1 = 1$

$$a_0 = \alpha_1 + \alpha_2$$

$$0 = \alpha_1 + \alpha_2$$

$$\alpha_1 = -\alpha_2$$

$$1 \alpha_1 = \left(\frac{1 + \sqrt{5}}{2}\right) \alpha_1 + \left(\frac{1 - \sqrt{5}}{2}\right) \alpha_2$$

$$1 = -\left(\frac{1 + \sqrt{5}}{2}\right) \alpha_2 + \left(\frac{1 - \sqrt{5}}{2}\right) \alpha_2$$

$$1 = -\frac{\alpha_2}{2} - \frac{\sqrt{5}\alpha_2}{2} + \frac{\alpha_2}{2} - \frac{\sqrt{5}\alpha_2}{2}$$

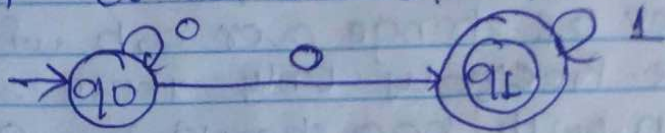
$$1 = -2 \cdot \frac{\sqrt{5}}{2} \alpha_2$$

$$\alpha_2 = -0.4472$$

$$\alpha_1 = 0.4472 \quad \#$$

Imp

Conversion of NFA to DFA:



state diagram.

let δ' be the transition defined for DFA,

$$\begin{aligned}\delta'(\{q_0\}, 0) &= \delta(\{q_0\}, 0) \\ &= \{q_0, q_1\} \\ &= \text{New state}\end{aligned}$$

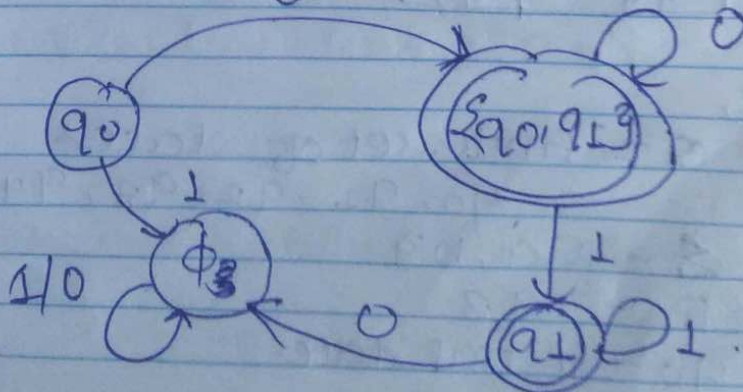
$$\delta'(\{q_0\}, 1) = \emptyset$$

$$\delta'(\{q_1\}, 0) = \emptyset$$

$$\delta'(\{q_1\}, 1) = \{q_1\}$$

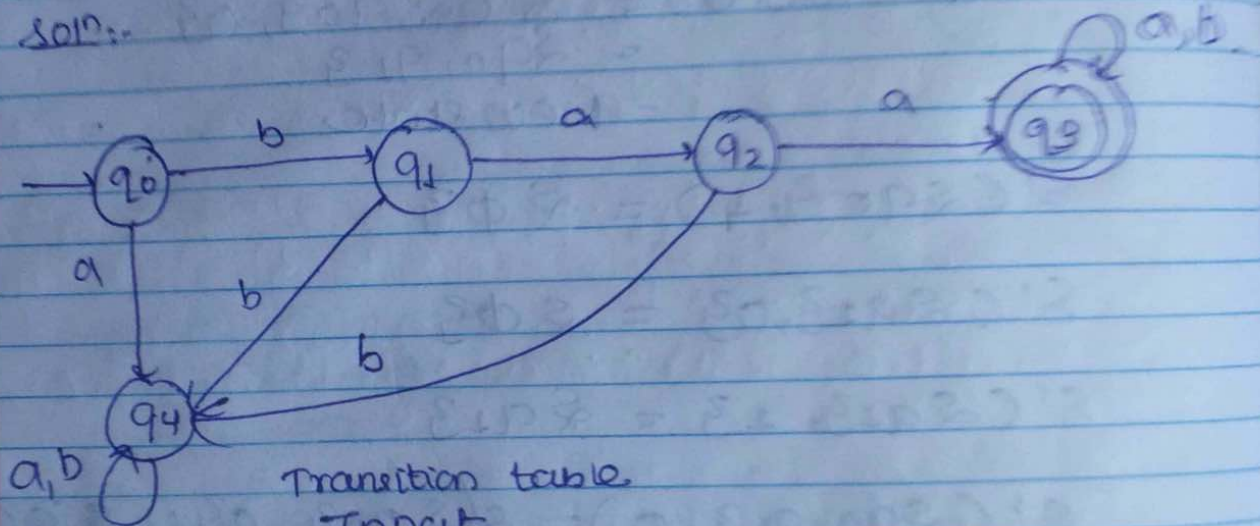
$$\begin{aligned}\delta'(\{q_0, q_1\}, 0) &= \delta(\{q_0\}, 0) \cup \delta(\{q_1\}, 0) \\ &= \{q_0, q_1\} \cup \emptyset \\ &= \{q_0, q_1\}\end{aligned}$$

$$\delta(\{q_0, q_1\}, 1) = \{q_1\}$$



Design a finite state automaton that accepts only those set of strings over a, b which starts with baa . Precisely only those strings which begin with baa should be accepted and other strings over (a, b) should be rejected. Use design should include the proper design of finite state automaton & transition table & transition diagram.

Soln:-



Transition table

Input		
Q	a	b
q0	q4	q1
q1	q2	q4
q2	q3	q4
q3	q3	q3
q4	q4	q4

Here,

Q = finite set of states
 $= q_0, q_1, q_2, q_3, q_4$

$\Sigma = \{a, b\}$

$F = q_3$

q_0 = initial state.

δ is a transition function defined by

$$\delta(q_0, a) \rightarrow q_4$$

$$\delta(q_0, b) \rightarrow q_1$$

$$\delta(q_1, a) \rightarrow q_2$$

$$\delta(q_1, b) \rightarrow q_4$$

$$\delta(q_2, a) \rightarrow q_3$$

$$\delta(q_2, b) \rightarrow q_4$$

$$\delta(q_3, a) \rightarrow q_3$$

$$\delta(q_3, b) \rightarrow q_3$$

$$\delta(q_4, a) \rightarrow q_4$$

$$\delta(q_4, b) \rightarrow q_4$$

design a finite state automata that accepts precisely those strings over $\{a, b\}$ that contain an even no. of a .

solⁿ:

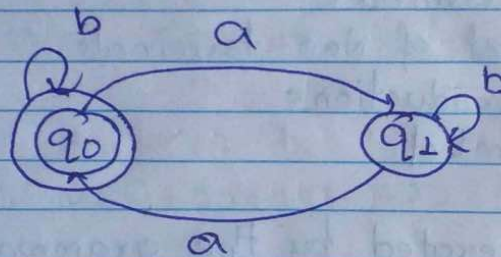
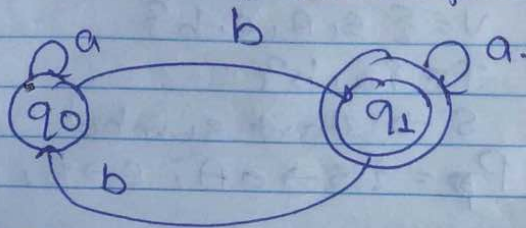
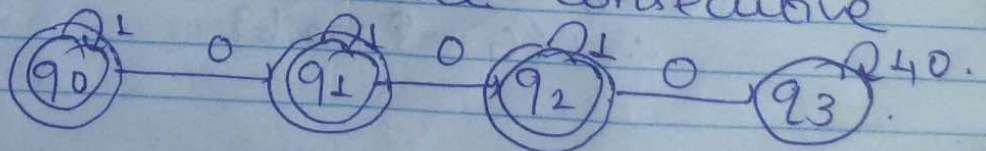


fig: state diagram.

that contains odd no- of b .



that do not contain three consecutive zeros.



Languages & Grammars

- Formal languages are specified by well defined set of syntax or rules used to model the natural language and to communicate with the computers.

Grammars for Formal Language:

- Grammar is denoted by $G = (V, T, P, S)$

Where

V = set of vocabulary (consisting of Non-terminal symbols, terminals symbols & start symbol)

T = set of Terminals

$N = V - T$ = set of Non-terminals

P = Set of productions

S = start symbol

- Language generated by the grammar is denoted by $L(G)$ is the set of string of terminals that are derivable from the start symbol.

Example:

Let $G = (V, T, P, S)$

where, $V = \{S, A, a, b\}$

$T = \{a, b\}$

$S \rightarrow$ start symbol

$P = \{S \rightarrow aA, S \rightarrow b, A \rightarrow aa\}$

Soln
 $S \rightarrow aA$
 $\rightarrow aaa$

$S \rightarrow b$

$\therefore L(G) = \{aaa, b\}$

a) Let G be the grammar with vocabulary $V = \{S, 0, 1\}$,

set of terminals $(T) = \{0, 1\}$

starting symbol $(S) \rightarrow S$

& production $(P) = \{S \rightarrow 11S, S \rightarrow 0\}$

$L(G) = ?$, derive the derivation tree for the string 11110.

Solⁿ:-

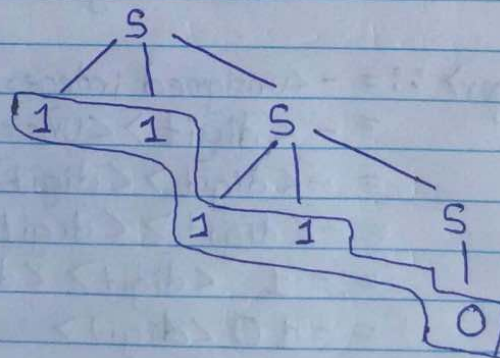
$S \rightarrow 0 \rightarrow 110$

$S \rightarrow 11S \rightarrow 110$

$\rightarrow 1111S \rightarrow 11110$

$\rightarrow 11111S \rightarrow 1111110$

$L(G) = \{0, 110, 11110, 1111110, \dots\}$



BNF (Backus Naur (Normal)) Form:

→ An alternative way of to denote the set of production
 $S ::= T$

→ In this form, the non-terminal symbol starts with "<" and ends with ">" symbol.

For example, the production of the form
 $S ::= T_1 \mid S ::= T_2 \mid S ::= T_3 \dots S ::= T_n$ can be
 combined as:
 $S ::= T_1 \mid T_2 \mid T_3 \dots \mid T_n$
 OR

BNF for integers:

$\langle \text{digit} \rangle ::= 0 \mid 1 \mid 2 \mid \dots \mid 9$

$\langle \text{integers} \rangle ::= \langle \text{unsigned integer} \rangle \mid \langle \text{signed integer} \rangle$

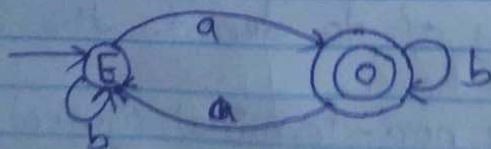
$\langle \text{signed integer} \rangle ::= + \langle \text{unsigned integer} \rangle \mid - \langle \text{unsigned integer} \rangle$

$\langle \text{unsigned integer} \rangle ::= \langle \text{digit} \rangle \mid \langle \text{digit} \rangle \langle \text{unsigned integer} \rangle$

Q) Give a BNF for ^{integer} ~~1-108~~ integer -108.

$\langle \text{signed integer} \rangle ::= - \langle \text{unsigned integer} \rangle$
 $= - \langle \text{digit} \rangle \langle \text{unsigned integer} \rangle$
 $= - \langle \text{digit} \rangle \langle \text{digit} \rangle \langle \text{unsigned integer} \rangle$
 $= - \langle \text{digit} \rangle \langle \text{digit} \rangle \langle \text{digit} \rangle$
 $= - 1 \langle \text{digit} \rangle \langle \text{digit} \rangle$
 $= - 10 \langle \text{digit} \rangle$
 $= - 108$

Relation between Grammar and FSA & vice-versa



Solⁿ Grammar $G = (V, T, P, S)$

where, $V = \{E, O, a, b\}$

$T = \{a, b\}$

$P = \{E \rightarrow aO, E \rightarrow bE, O \rightarrow bO, O \rightarrow aE\}$

$S = E$

Imp

Consider the regular grammar $G = (N, T, P, \sigma)$ where, N = set of non-terminal symbols which is equal to $\{\sigma, C\}$

T = set of terminal symbols which is equal to $\{a, b\}$

P is the set of production rules = $\{\sigma \rightarrow b\sigma, \sigma \rightarrow aC, C \rightarrow bC, C \rightarrow \lambda\}$ & σ being the start symbol.

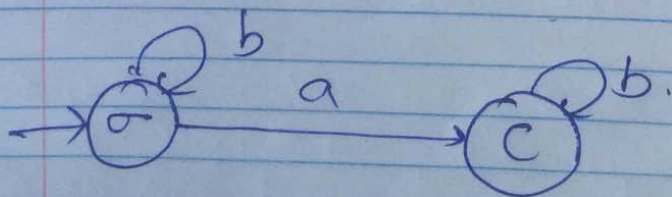
Construct a NDEFA equivalent to given regular grammar. Use this non-deterministic finite state automata (NDEFA) to generate equivalent DFA.

Solⁿ

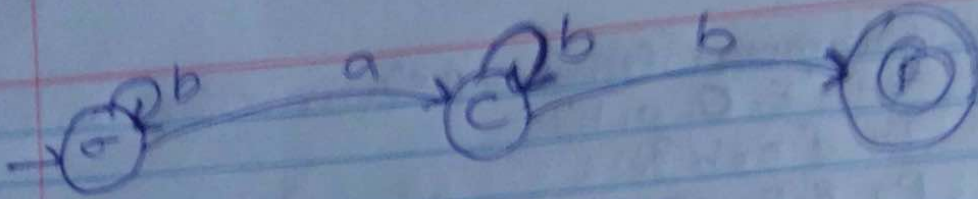
The production of the form $C \rightarrow \lambda$ can be partitioned into two productions of the form.

$C \rightarrow bF$

$F \rightarrow \lambda$



It is NFA as there is no unique transition.



Conversion to NFA DFA

